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J. Math. Anal. Appl. 317 (2006) 398–428
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Asymptotically optimal production policies in dynamic stochastic jobshops with limited buffers ☆

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Received 6 February 2004

Available online 17 February 2006

Submitted by J.A. Filar

Abstract

We consider a production planning problem for a jobshop with unreliable machines producing a number of products. There are upper and lower bounds on intermediate parts and an upper bound on finished parts. The machine capacities are modelled as finite state Markov chains. The objective is to choose the rate of production so as to minimize the total discounted cost of inventory and production. Finding an optimal control policy for this problem is difficult. Instead, we derive an asymptotic approximation by letting the rates of change of the machine states approach infinity. The asymptotic analysis leads to a limiting problem in which the stochastic machine capacities are replaced by their equilibrium mean capacities. The value function for the original problem is shown to converge to the value function of the limiting problem. The convergence rate of the value function together with the error estimate for the constructed asymptotic optimal production policies are established.

Keywords: Optimal production policy; Stochastic manufacturing systems; Stochastic dynamic programming; Discounted cost; Asymptotic analysis

☆ The research is supported in part by a faculty research grant from School of Management, The University of Texas at Dallas, a Distinguished Young Investigator Grant from the National Natural Sciences Foundation of China, and a grant from the Hundred Talents Program of the Chinese Academy of Sciences.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.01.034

1. Introduction

We consider a manufacturing system producing a variety of products in demand using machines in a general network configuration, which generalizes both the parallel and tandem machine models. Each product follows a process plan that specifies the sequence of machines it must visit and the operations performed by them. A process plan may call for multiple visits to a given machine, as is the case in semiconductor manufacturing (Lou and Kager [2]). Often the machines are unreliable. Over time they break down and must be repaired. A manufacturing system so described will be termed a *dynamic jobshop*. The term will be made mathematically precise in the next section.

It is in the nature of such a dynamic jobshop that the inventory of intermediate parts must remain nonnegative. In reality, the buffer capacities are also limited implying upper bounds on both intermediate as well as finished parts.

The problem under consideration is to control the production rates of intermediate parts and of finished parts in a manufacturing system consisting of a network of failure-prone machines. The objective is to meet the demand for finished products at the minimum possible discounted cost of production, inventories, and backlogs. The problem is not easy to solve, particularly because of the state constraints associated with the intermediate and finished parts. Certainly, no explicit solution, unlike in a single machine system without any state constraints (Akella and Kumar [1]), is available for our problem.

Recognizing the complexity of the problem, Sethi and Zhou [7] develop a hierarchical approach for approximately solving the stochastic optimal production planning problem of a jobshop with a discount cost criterion, when the rates of changes in machine states are much larger than the rate of discounting of costs (see also Sethi and Zhang [5]). Sethi and Zhou show that the problem can be approximated by a deterministic optimization problem. Then by using the optimal control of the deterministic problem, they construct a production policy for the original stochastic problem, which is asymptotically optimal as the rate of changes in machine states approaches infinity. However, this paper did not consider the upper bound constraints associated with intermediate and finished parts. Moreover, their method of constructing asymptotic optimal production policies does not apply to systems with upper bound state constraints.

Sethi, Zhang and Zhang [4] consider an N-machine flowshop with limited buffers. They develop a method of *shrinking*, *entire lifting*, and *modification* to construct an asymptotic optimal open-loop production policy for the N-machine flowshop from a near-optimal production policy for the corresponding limiting problem. Here "shrinking" means to find a production policy that uses a little less than the full machine capacities at a time. Based on this shrinking, the entire lifting procedure involves an appropriate increase in production rates in some intervals so as to make the inventory levels in the internal buffers to be positive. After shrinking and entire lifting, they obtain a near-optimal production policy for the limiting problem. Using this near-optimal production policy, they construct a production policy for the original problem, which they further modify to make it also a near-optimal admissible production policy for the original problem.

The purpose of this paper is to obtain asymptotic optimal policies for a jobshop with finite buffers. Owing to the fact that more than one machine can feed a given buffer, the shrinking, entire lifting, and modification method of Sethi, Zhang and Zhang [4] is no longer sufficient in constructing asymptotic optimal production policies. To overcome this difficulty, we introduce a scheme of parameter distribution of the machine capacities. We also show that the control constructed by this technique is indeed near-optimal. The error estimate that we obtain is of the same order as the one obtained in Sethi, Zhang and Zhang [4].

The plan of our paper is as follows. In Section 2 we introduce the problem and specify the required assumptions. Section 3 is devoted to formulating the limiting control problem. In Section 4, we establish the convergence of the minimum discounted expected cost for the original problem to the minimum discounted cost for the limiting problem. Section 5 is devoted to constructing asymptotic optimal controls by using the results developed in Section 4. Finally in Section 6, we conclude the paper.

2. Problem formulation

We begin with a manufacturing system that consists of m_c failure-prone machines and n buffers including m internal buffers. We use the notation of the jobshop model developed in Sethi and Zhou [7], and Presman, Sethi and Suo [3]. Then we give a simple example to illustrate the model.

Let $\boldsymbol{k}(\varepsilon,t)=(k_1(\varepsilon,t),\dots,k_{m_c}(\varepsilon,t))$ denote a stochastic process defined on the standard probability space $(\Omega,\mathcal{F},\Pr)$ with $k_\ell(\varepsilon,t)$ representing the capacity of the ℓth machine at time t, $\ell=1,\dots,m_c$, where ε is a small parameter to be precisely specified later.

We denote the surplus at time t in buffer j by $x_j(\varepsilon,t)$. We write it in vector form as

$$\boldsymbol{x}(\varepsilon,t)=\big(x_1(\varepsilon,t),\dots,x_n(\varepsilon,t)\big)'.$$

Note that $x_j(\varepsilon,t)$, $j=1,2,\dots,m$, is called a work-in-process at time t and $x_j(\varepsilon,t)$, $j=m+1,\dots,n$, is called a surplus of the finished product j at time t.

Let (V,A) denote a manufacturing digraph, where V is a finite nonempty set of vertices and A is the collection of ordered arcs. Let us now suppose that (V,A) contains a total of (n_0+n+1) vertices including n_0 sources, the sink, m internal buffers, and $(n-m)$ external buffers for some integer m and n with $0\leqslant m\leqslant n-1$ and $n\geqslant 1$. Let $\mathcal{K}=\{K_1,K_2,\dots,K_{m_c}\}$ denotes the corresponding placement, i.e., $\mathcal{K}$ is a partition of $B=\{(i,j)\in A\colon i\leqslant m\}$, namely, $\emptyset\neq K_j\subset B$, $K_j\cap K_\ell=\emptyset$ for $j\neq\ell$, and $\bigcup_{j=1}^{m_c}K_j=B$; see Sethi and Zhou [7] or Sethi and Zhang [5] for further details. The controls $u_{i,j}(\varepsilon,t)$ with $(i,j)\in K_\ell$, $\ell=1,\dots,m_c$, $t\geqslant 0$, must satisfy the following constraints:

$$0\leqslant\sum_{(i,j)\in K_\ell}u_{i,j}(\varepsilon,t)\leqslant k_\ell(\varepsilon,t)\quad\text{for all } t\geqslant 0,\ \ell=1,\dots,m_c. \tag{1}$$

Then the dynamics of the system are given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x_j(\varepsilon,t)&=\sum_{\ell=-n_0+1}^{j-1}u_{\ell,j}(\varepsilon,t)-\sum_{\ell=j+1}^{n}u_{j,\ell}(\varepsilon,t),\quad 1\leqslant j\leqslant m,\\
\frac{\mathrm{d}}{\mathrm{d}t}x_j(\varepsilon,t)&=\sum_{\ell=-n_0+1}^{m}u_{\ell,j}(\varepsilon,t)-z_j,\quad m+1\leqslant j\leqslant n,
\end{aligned}\tag{2}$$

with $\boldsymbol{x}(\varepsilon,0)=(x_1(\varepsilon,0),\dots,x_n(\varepsilon,0))'=(x_1,\dots,x_n)'=\boldsymbol{x}$, where z_j is the demand rate for the finished product j, and $u_{j,\ell}(\varepsilon,t)=0$ if $(j,\ell)\notin A$. The state constraints are

$$\begin{aligned}
&0\leqslant x_j(\varepsilon,t)\leqslant H_j,\quad t\geqslant 0,\ j=1,\dots,m,\\
&-\infty<x_j(\varepsilon,t)\leqslant H_j,\quad t\geqslant 0,\ j=m+1,\dots,n,
\end{aligned}\tag{3}$$

where H_j is the capacity of buffer j. As a capacity measure, of course, we assume that $H_j>0$. Note that if $x_j(\varepsilon,t)>0$, $j=1,\dots,n$, we have an inventory in buffer j, and if $x_j(\varepsilon,t)<0$, $j=m+1,\dots,n$, we have a shortage of the finished product j.

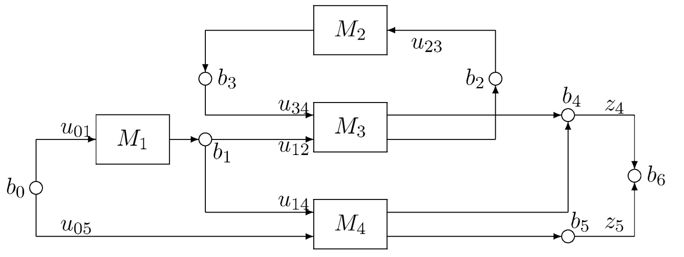

Fig. 1. A typical dynamic jobshop.

Example 2.1. To illustrate the model, let us consider a simple jobshop given in Fig. 1.

We have four machines M_1, M_2, M_3 and M_4 and seven buffers $b_0, b_1, \dots, b_6$ (or simply buffers $0, 1, \dots, 6$). Machine M_i $(i = 1, 2, 3, 4)$ has the production capacity of $k_i(\varepsilon, t)$ at time t. Buffers b_0 and b_6 are fictitious in the sense that b_0 is an infinite source containing all required raw materials and b_6 is a sink with no constraints. Buffers b_4 and b_5 are external buffers corresponding to specific final products and the remaining buffers are internal buffers corresponding to specific intermediate products. Quantities z_4 and z_5 represent the rates of demand of the final products 4 and 5, respectively, $u_{i,j}(t)$ is the rate of conversion from items in buffer i to items in buffer j by using the machine on the arc (i, j), $i = 0, 1, 2, 3$ and $j = 1, 2, 3, 4, 5$. Let $x_i(t)$, $i = 1, \dots, 5$, be the contents of buffer b_i at time t. Clearly $x_i(t)$, $i = 1, 2, 3$, cannot become negative, and for $i = 4, 5$, $x_i(t)$ defines the inventory/shortage of final product i and it may take negative values.

The system dynamics and the constraints are stated below.

System dynamics:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = u_{01}(t) - u_{12}(t) - u_{14}(t), \qquad && \frac{\mathrm{d}}{\mathrm{d}t}x_4(t) = u_{14}(t) + u_{34}(t) - z_4,\\
&\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = u_{12}(t) - u_{23}(t), && \frac{\mathrm{d}}{\mathrm{d}t}x_5(t) = u_{05}(t) - z_5.\\
&\frac{\mathrm{d}}{\mathrm{d}t}x_3(t) = u_{23}(t) - u_{34}(t), &&
\end{aligned} \tag{4}$$

State constraints:

$$0 \leqslant x_i(t) \leqslant H_i, \quad i = 1, 2, 3, \quad \text{and} \quad -\infty < x_i(t) \leqslant H_i,\ i = 4, 5. \tag{5}$$

Control constraints:

$$\begin{aligned}
&u_{01}(t) \leqslant k_1(\varepsilon, t), && u_{05}(t) + u_{14}(t) \leqslant k_4(\varepsilon, t),\\
&u_{12}(t) + u_{34}(\varepsilon, t) \leqslant k_3(\varepsilon, t), \qquad && u_{23}(t) \leqslant k_2(\varepsilon, t),
\end{aligned} \tag{6}$$

where H_i is the size of the buffer $i, i = 1, 2, 3, 4, 5$. In this example, the placements for the jobshop in Fig. 1 are $K_1 = \{(0, 1)\}$, $K_2 = \{(2, 3)\}$, $K_3 = \{(3, 4), (1, 2)\}$, $K_4 = \{(0, 5), (1, 4)\}$.

It is convenient to write the control in a vector form. To do this, for $\ell = -n_0 + 1, \dots, 0$ and $j = 1, \dots, m$, let

$$\boldsymbol{u}_\ell(\varepsilon, t) = \begin{pmatrix} u_{\ell,1}(\varepsilon, t) \\ \vdots \\ u_{\ell,n}(\varepsilon, t) \end{pmatrix}, \qquad \boldsymbol{u}^o_j(\varepsilon, t) = \begin{pmatrix} u_{j,j+1}(\varepsilon, t) \\ \vdots \\ u_{j,n}(\varepsilon, t) \end{pmatrix}, \tag{7}$$

$$\boldsymbol{u}(\varepsilon,t)=\begin{pmatrix}\boldsymbol{u}_{-n_0+1}(\varepsilon,t)\\ \vdots\\ \boldsymbol{u}_0(\varepsilon,t)\\ \boldsymbol{u}_1^o(\varepsilon,t)\\ \vdots\\ \boldsymbol{u}_m^o(\varepsilon,t)\end{pmatrix},\quad\text{and}\quad \boldsymbol{z}=\begin{pmatrix}z_{m+1}\\ \vdots\\ z_n\end{pmatrix}. \tag{8}$$

Then the system equation in (2) can be written in the following vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(\varepsilon,t)=(A_{-n_0+1},\ldots,A_{m+1})\begin{pmatrix}\boldsymbol{u}(\varepsilon,t)\\ \boldsymbol{z}\end{pmatrix},\qquad \boldsymbol{x}(\varepsilon,0)=\boldsymbol{x}, \tag{9}$$

with a suitable choice of $(A_{-n_0+1},\ldots,A_{m+1})$, which is an $n\times\tilde{n}$ matrix with $\tilde{n}=n_0n+(n-m)+\sum_{\ell=1}^{m}(n-\ell)$. Let $\mathcal{I}$ denote the state space, i.e., $\mathcal{I}=\prod_{\ell=1}^{m}[0,\ H_\ell]\times\prod_{\ell=m+1}^{n}(-\infty,\ H_\ell]$. Furthermore, we introduce a linear operator $\mathcal{L}$ from $\Re_+^{\tilde{n}}$ to $\mathcal{I}$:

$$\mathcal{L}\big(\boldsymbol{u}(\varepsilon,t),\boldsymbol{z}\big)=(A_{-n_0+1},\ldots,A_{m+1})\begin{pmatrix}\boldsymbol{u}(\varepsilon,t)\\ \boldsymbol{z}\end{pmatrix}.$$

Then, (9) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(\varepsilon,t)=\mathcal{L}\big(\boldsymbol{u}(\varepsilon,t),\boldsymbol{z}\big),\qquad \boldsymbol{x}(\varepsilon,0)=\boldsymbol{x}.$$

We are now in a position to formulate the stochastic optimal control problem for our jobshop defined by (1)–(3). Let $\hat{n}=n_0n+\sum_{\ell=1}^{m}(n-\ell)$ and $\mathcal{U}=\Re_+^{\hat{n}}$. For $\boldsymbol{k}=(k_1,\ldots,k_{m_c})$, let

$$\mathcal{U}(\boldsymbol{k})=\Bigg\{(u_{i,\ell}):\ (u_{i,\ell})\in\mathcal{U},\ 0\leqslant\sum_{(i,\ell)\in K_j}u_{i,\ell}\leqslant k_j,\ 1\leqslant j\leqslant m_c\Bigg\}.$$

By (7) and (8), for each $(u_{i,j})\in\mathcal{U}(\boldsymbol{k})$, we can generate a unique nonnegative $\tilde{n}$-dimensional vector $\boldsymbol{u}$. In the rest of the paper, we use $\boldsymbol{u}$ and $(u_{i,j})$ $(\in\mathcal{U}(\boldsymbol{k}))$ interchangeably. For $\boldsymbol{x}\in\mathcal{I}$ and $\boldsymbol{k}$,

$$\begin{aligned}\mathcal{U}(\boldsymbol{x},\boldsymbol{k})=\Bigg\{\boldsymbol{u}:\ \boldsymbol{u}\in\mathcal{U}(\boldsymbol{k})\text{ and }x_j=0\ &\Rightarrow\sum_{i=-n_0+1}^{j-1}u_{i,j}-\sum_{i=j+1}^{n}u_{j,i}\geqslant 0,\\ \text{and }x_j=H_j\ &\Rightarrow\sum_{i=-n_0+1}^{j-1}u_{i,j}-\sum_{i=j+1}^{n}u_{j,i}\leqslant 0,\ j=1,\ldots,m,\\ x_j=H_j\ &\Rightarrow\sum_{i=-n_0+1}^{j-1}u_{i,j}-z_j\leqslant 0,\ j=m+1,\ldots,n\Bigg\}.\end{aligned}$$

Definition 2.1. We say that a control $\boldsymbol{u}(\varepsilon,\cdot)$ is *admissible* with respect to the initial state vector $\boldsymbol{x}=(x_1,\ldots,x_n)'\in\mathcal{I}$ and $\boldsymbol{k}(0)=\boldsymbol{k}$ if

(i) $\boldsymbol{u}(\varepsilon,\cdot)$ is an $\mathcal{F}_{\varepsilon,t}$-adapted measurable process with $\mathcal{F}_{\varepsilon,t}=\sigma\{\boldsymbol{k}(\varepsilon,s):\ 0\leqslant s\leqslant t\}$,
(ii) $\boldsymbol{u}(\varepsilon,t)\in\mathcal{U}(\boldsymbol{k}(\varepsilon,t))$ for all $t\geqslant 0$, and
(iii) the corresponding state process $\boldsymbol{x}(\varepsilon,t)=(x_1(\varepsilon,t),\ldots,x_n(\varepsilon,t))'\in\mathcal{I}$ for all $t\geqslant 0$.

We assume the following conditions on the random process $\boldsymbol{k}(\varepsilon,t)$ and the cost function $h(\cdot)$ and $c(\cdot)$ throughout this paper:

(A1) Let $\mathcal{M} = \{\boldsymbol{k}^1, \ldots, \boldsymbol{k}^p\}$ for some given integer $p \geqslant 1$, where $\boldsymbol{k}^j = (k_1^j, \ldots, k_{m_c}^j)$, with k_ℓ^j, $\ell = 1, \ldots, m_c$, denoting the capacity of the ℓth machine, $j = 1, \ldots, p$. The capacity process $\boldsymbol{k}(\varepsilon, t) \in \mathcal{M}$ is a finite state Markov chain with the infinitesimal generator $Q = Q^{(1)} + \varepsilon^{-1} Q^{(2)}$, where $Q^{(1)} = (q_{ij}^{(1)})$ and $Q^{(2)} = (q_{ij}^{(2)})$ are matrices such that $q_{ij}^{(r)} \geqslant 0$ if $j \neq i$, and $q_{ii}^{(r)} = -\sum_{j \neq i} q_{ij}^{(r)}$ for $r = 1, 2$. Moreover, $Q^{(2)}$ is irreducible and, without any loss of generality, it is taken to be the one that satisfies

$$\min_{ij}\left\{\left|q_{ij}^{(2)}\right|: q_{ij}^{(2)} \neq 0\right\} = 1.$$

Let $\gamma = (\gamma_1, \ldots, \gamma_p)$ denote the equilibrium distribution of $Q^{(2)}$. That is, γ is the only nonnegative solution to the equation

$$\gamma Q^{(2)} = 0 \quad \text{and} \quad \sum_{i=1}^{p} \gamma_i = 1. \tag{10}$$

(A2) $h(\cdot)$ and $c(\cdot)$ are nonnegative convex functions. For all $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{I}$ and $\boldsymbol{u}, \boldsymbol{u}' \in U(\boldsymbol{k}^j)$, $j = 1, \ldots, p$, there exist constants C_0 and $K_h \geqslant 1$ such that

$$\left|h(\boldsymbol{x}) - h(\boldsymbol{x}')\right| \leqslant C_0\left(1 + |\boldsymbol{x}|^{K_h} + |\boldsymbol{x}'|^{K_h}\right)|\boldsymbol{x} - \boldsymbol{x}'|$$

and

$$\left|c(\boldsymbol{u}) - c(\boldsymbol{u}')\right| \leqslant C_0|\boldsymbol{u} - \boldsymbol{u}'|.$$

Remark 2.1. In Assumption (A1), the small parameter $\varepsilon > 0$ serves as the time scale factor. The smaller the ε, the more rapidly the process $\boldsymbol{k}(\varepsilon, \cdot)$ jumps in $\mathcal{M}$. Additional interpretations of this setup and practical considerations can be found in Sethi and Zhang [5, Chapter 5]; see also Yin and Zhang [8] for multi-block decomposition of generator Q^ε.

We use $\mathcal{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{k})$ to denote the set of all admissible controls with respect to $\boldsymbol{x} \in \mathcal{I}$ and $\boldsymbol{k}(\varepsilon, 0) = \boldsymbol{k}$. We use $\mathcal{P}^\varepsilon$ to denote our control problem, i.e.,

$$\mathcal{P}^\varepsilon: \begin{cases} \text{minimize} \quad J^\varepsilon\big(\boldsymbol{x}, \boldsymbol{u}(\varepsilon, \cdot), \boldsymbol{k}\big) = \mathsf{E} \displaystyle\int_0^\infty e^{-\rho t}\big[h\big(\boldsymbol{x}(\varepsilon, t)\big) + c\big(\boldsymbol{u}(\varepsilon, t)\big)\big]\,\mathrm{d}t, \\ \text{subject to} \quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(\varepsilon, t) = \mathcal{L}\big(\boldsymbol{u}(\varepsilon, t), z\big), \ \boldsymbol{x}(\varepsilon, 0) = \boldsymbol{x}, \\ \boldsymbol{u}(\varepsilon, \cdot) \in \mathcal{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{k}) \end{cases} \\ \text{value function} \quad V^\varepsilon(\boldsymbol{x}, \boldsymbol{k}) = \displaystyle\inf_{\boldsymbol{u}(\varepsilon, \cdot) \in \mathcal{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{k})} J^\varepsilon\big(\boldsymbol{x}, \boldsymbol{u}(\varepsilon, \cdot), \boldsymbol{k}\big). \end{cases} \tag{11}$$

3. The limiting control problem

In this section we derive the limiting control problem as $\varepsilon \to 0$. Intuitively, as the rates of the machine breakdown and repair approach infinity, the problem $\mathcal{P}^\epsilon$, which is termed the *original problem*, can be approximated by a simpler problem called the *limiting problem*, where the stochastic machine capacity process $\boldsymbol{k}(\varepsilon, t)$ is replaced by a weighted form.

The Hamilton–Jacobi–Bellman equation in the directional derivative (HJBDD) sense with the discounted optimal control problem in $\mathcal{P}^\varepsilon$, as shown in Sethi, Zhang and Zhang [4], takes the form

$$V^\varepsilon\big(\boldsymbol{x},\boldsymbol{k}^r\big)=\inf_{\boldsymbol{u}\in U(\boldsymbol{x},\boldsymbol{k}^r)}\left\{\frac{\partial V^\varepsilon(\boldsymbol{x},\boldsymbol{k}^r)}{\partial\mathcal{L}(\boldsymbol{u},z)}+c(\boldsymbol{u})\right\}+h(\boldsymbol{x})$$
$$+\left(Q^{(1)}+\frac{1}{\varepsilon}Q^{(2)}\right)V^\varepsilon(\boldsymbol{x},\cdot)\big(\boldsymbol{k}^r\big), \tag{12}$$

where $\frac{\partial V^\varepsilon(\boldsymbol{x},\boldsymbol{k}^r)}{\partial\mathcal{L}(\boldsymbol{u},z)}$ denotes the directional derivative of $V^\varepsilon(\boldsymbol{x},\boldsymbol{k}^r)$ along the direction $\mathcal{L}(\boldsymbol{u},z)$, and $Qf(\cdot)(\boldsymbol{k}^r):=\sum_{i\neq r}q_{ri}(f(\boldsymbol{k}^i)-f(\boldsymbol{k}^r))$ for any function $f(\cdot)$ on $\mathcal{M}$. Moreover, Sethi and Zhang [5] show that $V^\varepsilon(\boldsymbol{x},\boldsymbol{k}^r)$ is a solution of (12).

The limiting problem can be formulated as follows. Consider an augmented control

$$U(\cdot)=\big(\boldsymbol{u}^1(\cdot),\dots,\boldsymbol{u}^p(\cdot)\big),$$

where $\boldsymbol{u}^i(t)\in\mathcal{U}(\boldsymbol{k}^i)$ and $\boldsymbol{u}^i(t)$, $t\geqslant 0$, is a deterministic process.

Definition 3.1. For $\boldsymbol{x}\in\mathcal{I}$, let $\mathcal{A}^0(\boldsymbol{x})$ denote the set of measurable controls

$$U(\cdot)=\big(\boldsymbol{u}^1(\cdot),\dots,\boldsymbol{u}^p(\cdot)\big),$$

such that the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)=\mathcal{L}\left(\sum_{i=1}^p\gamma_i\boldsymbol{u}^i(t),z\right),\qquad \boldsymbol{x}(0)=\boldsymbol{x},$$

satisfies $\boldsymbol{x}(t)\in\mathcal{I}$ for all $t\geqslant 0$.

The objective of the problem is to choose a control $U(\cdot)\in\mathcal{A}^0(\boldsymbol{x})$ that minimizes

$$J\big(\boldsymbol{x},U(\cdot)\big)=\int_0^\infty e^{-\rho t}\left[h\big(\boldsymbol{x}(s)\big)+\sum_{i=1}^p\gamma_i c\big(\boldsymbol{u}^i(s)\big)\right]\mathrm{d}s.$$

We use $\mathcal{P}^0$ to denote this problem, known as the limiting control problem, and restate it as follows:

$$\mathcal{P}^0:\quad\begin{cases}\text{minimize}\quad J\big(\boldsymbol{x},U(\cdot)\big)=\displaystyle\int_0^\infty e^{-\rho t}\left[h\big(\boldsymbol{x}(s)\big)+\sum_{i=1}^p\gamma_i c\big(\boldsymbol{u}^i(s)\big)\right]\mathrm{d}s,\\ \text{subject to}\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t)=\mathcal{L}\left(\displaystyle\sum_{i=1}^p\gamma_i\boldsymbol{u}^i(t),z\right),\boldsymbol{x}(0)=\boldsymbol{x},\quad U(\cdot)\in\mathcal{A}^0(\boldsymbol{x}),\\ \text{value function}\quad V(\boldsymbol{x})=\displaystyle\inf_{U(\cdot)\in\mathcal{A}^0(\boldsymbol{x})}J\big(\boldsymbol{x},U(\cdot)\big).\end{cases} \tag{13}$$

The HJBDD associated with $\mathcal{P}^0$ is

$$V(\boldsymbol{x})=\inf_{U\in\mathcal{U}^0(\boldsymbol{x})}\left\{\frac{\partial V(\boldsymbol{x})}{\partial\mathcal{L}(\sum_{i=1}^p\gamma_i\boldsymbol{u}^i,z)}+\sum_{i=1}^p\gamma_i c\big(\boldsymbol{u}^i\big)\right\}+h(\boldsymbol{x}), \tag{14}$$

where

$$\mathcal{U}^0(\boldsymbol{x}) = \Bigg\{ (\boldsymbol{u}^1, \ldots, \boldsymbol{u}^p)\colon \boldsymbol{u}^i \in \mathcal{U}(\boldsymbol{k}^i) \text{ and } x_j = 0$$

$$\Rightarrow \sum_{i=1}^{p} \sum_{\ell=-n_0+1}^{j-1} \gamma_i u^i_{\ell,j} - \sum_{i=1}^{p} \sum_{\ell=j+1}^{n} \gamma_i u^i_{j,\ell} \geqslant 0,$$

$$x_j = H_j \Rightarrow \sum_{i=1}^{p} \sum_{\ell=-n_0+1}^{j-1} \gamma_i u^i_{\ell,j} - \sum_{i=1}^{p} \sum_{\ell=j+1}^{n} \gamma_i u^i_{j,\ell} \leqslant 0,\ j = 1, \ldots, m,$$

$$\text{and } x_j = H_j \Rightarrow \sum_{i=1}^{p} \sum_{\ell=-n_0+1}^{m} \gamma_i u^i_{\ell,j} - z_j \leqslant 0,\ j = m+1, \ldots, n \Bigg\}.$$

4. Convergence of value functions

In this section we consider the convergence of the value function (minimum expected cost) $V^\varepsilon(\boldsymbol{x}, \boldsymbol{k})$ as ε goes to zero, and establish its convergence rate. First we give without proof the following lemma similar to Lemma C.3 of Sethi and Zhang [5].

Lemma 4.1. *Let $P(\varepsilon, t)$ denote the transition matrix of the Markov process $\boldsymbol{k}(\varepsilon, \cdot)$. Then*

$$\left| P(\varepsilon, t) - \bar{P} \right| \leqslant C_1 \left(\varepsilon + e^{-\kappa t/\varepsilon} \right),$$

for some positive constant C_1 and κ, where $\bar{P} = (\gamma_1 \mathbf{1}, \ldots, \gamma_p \mathbf{1})$ with $\mathbf{1} = (1, \ldots, 1)'$ and $\gamma = (\gamma_1, \ldots, \gamma_p)$ given in (10). *Moreover, for all $\boldsymbol{k}^r \in \mathcal{M}$ and $t \geqslant 0$,*

$$\left| \Pr\{ \boldsymbol{k}(\varepsilon, t) = \boldsymbol{k}^r \} - \gamma_r \right| \leqslant C_1 \left(\varepsilon + e^{-\kappa t/\varepsilon} \right).$$

The next result we require is as follows:

Lemma 4.2. *Let*

$$\Phi(\varepsilon, t) = \Phi\big(\boldsymbol{k}(\varepsilon, t)\big) = \big(I_{\{\boldsymbol{k}(\varepsilon,t)=\boldsymbol{k}^1\}}, \ldots, I_{\{\boldsymbol{k}(\varepsilon,t)=\boldsymbol{k}^p\}} \big)'.$$

Then for any bounded deterministic measurable process $\beta(\cdot)$, $\delta \in (0, \frac{1}{2})$, and τ, which is a Markov time with respect to $\boldsymbol{k}(\varepsilon, \cdot)$, there exist positive constants C_2, ε_0 and κ_0 such that for all $T \geqslant 0$ and $\varepsilon \in (0, \varepsilon_0)$,

$$\Pr\left(\sup_{0 \leqslant t \leqslant T} \left| \int_{\tau}^{\tau+t} \left[\Phi(\varepsilon, s) - \gamma' \right] \beta(s)\, \mathrm{d}s \right| \geqslant \varepsilon^\delta \right) \leqslant C_2 e^{-\kappa_0 \varepsilon^{-(1/2-\delta)} (1+T)^{-3}}.$$

Proof. The proof is similar to Corollary C.4 of Sethi and Zhang [5]. Here we omit the proof. □

In order to get the required convergence result, we need the following auxiliary lemma, which is a key step towards our main result.

Lemma 4.3. *For any $\delta \in (0, \frac{1}{2})$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}$, there exist positive constants C and $\varepsilon_0 \leqslant 1$, $\boldsymbol{x}(\delta) = (x_1(\delta), \ldots, x_n(\delta)) \in \mathcal{I}$, and control for the limiting* (*deterministic*) *problem*

$$U(\delta, \varepsilon, \cdot) = \big(\boldsymbol{u}^1(\delta, \varepsilon, \cdot), \ldots, \boldsymbol{u}^p(\delta, \varepsilon, \cdot)\big) \in \mathcal{A}^0(\boldsymbol{x}),$$

such that for all $\varepsilon \in (0,\ \varepsilon_0)$,

$$\big|\boldsymbol{x} - \boldsymbol{x}(\delta)\big| \leqslant C\big(1 + |\boldsymbol{x}|\big)\varepsilon^{\delta}. \tag{15}$$

In addition, let $\boldsymbol{x}(\delta, \varepsilon, t)$ denote the trajectory under $U(\delta, \varepsilon, t)$ with $\boldsymbol{x}(\delta, \varepsilon, 0) = \boldsymbol{x}(\delta)$. Then,

$$\inf_{0 \leqslant s < \infty} x_j(\delta, \varepsilon, s) \geqslant \varepsilon^{\delta}, \quad j = 1, \ldots, m, \tag{16}$$

$$\sup_{0 \leqslant s < \infty} x_j(\delta, \varepsilon, s) \leqslant H_j - \varepsilon^{\delta}, \quad j = 1, \ldots, n, \tag{17}$$

and

$$V(\boldsymbol{x}) + C\big(1 + |\boldsymbol{x}|^{K_h+1}\big)\varepsilon^{\delta} > \int_0^{\infty} \left[h\big(\boldsymbol{x}(\delta, \varepsilon, t)\big) + \sum_{j=1}^{p} \gamma_j c\big(\boldsymbol{u}^j(\delta, \varepsilon, t)\big) \right] \mathrm{d}t, \tag{18}$$

where K_h is given in Assumption (A2).

Proof. For each fixed $\varepsilon > 0$ and $\boldsymbol{x} \in \mathcal{I}$, we select

$$\widetilde{U}(\cdot) = \big(\tilde{\boldsymbol{u}}^1(\cdot), \ldots, \tilde{\boldsymbol{u}}^p(\cdot)\big) \in \mathcal{A}^0(\boldsymbol{x}) \tag{19}$$

to be an ε-optimal control for $\mathcal{P}^0$, i.e.,

$$\left| \int_0^{\infty} e^{-\rho t} \left[h\big(\tilde{\boldsymbol{x}}(t)\big) + \sum_{i=1}^{p} \gamma_i c\big(\tilde{\boldsymbol{u}}^i(t)\big) \right] \mathrm{d}t - V(\boldsymbol{x}) \right| \leqslant \varepsilon, \tag{20}$$

where $\tilde{\boldsymbol{x}}(t)$ is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \mathcal{L}\left(\sum_{i=1}^{p} \gamma_i \tilde{\boldsymbol{u}}^i(t), z \right), \quad \boldsymbol{x}(0) = \boldsymbol{x}.$$

Furthermore, let

$$a(H) = \max_{1 \leqslant j \leqslant n} \left\{ \frac{H_j}{H_j - 2\varepsilon^{\delta}} \right\}. \tag{21}$$

Define

$$\hat{\boldsymbol{u}}^i(t) = \frac{\tilde{\boldsymbol{u}}^i(t)}{a(H)}, \quad i = 1, \ldots, p,$$

and

$$\widehat{U}(\cdot) = \big(\hat{\boldsymbol{u}}^1(\cdot), \ldots, \hat{\boldsymbol{u}}^p(\cdot)\big).$$

Let

$$\hat{x}_j(t) = \frac{x_j}{a(H)} + \int_0^t \left[\sum_{\ell=-n_0+1}^{j-1} \hat{u}_{\ell,j}(s) - \sum_{\ell=j+1}^{n} \hat{u}_{j,\ell}(s) \right] \mathrm{d}s, \quad j = 1, \ldots, m,$$

$$\hat{x}_j(t) = \frac{x_j}{a(H)} + \int_0^t \left[\sum_{\ell=-n_0+1}^{m} \hat{u}_{\ell,j}(s) - z_j \right] \mathrm{d}s, \quad j = m+1, \ldots, n.$$

Then,

$$\hat{x}_j(t) = \frac{\tilde{x}_j(t)}{a(H)}, \quad j = 1, \ldots, m, \tag{22}$$

$$\hat{x}_j(t) = \frac{\tilde{x}_j(t)}{a(H)} + \left(\frac{1}{a(H)} - 1 \right) z_j t, \quad j = m+1, \ldots, n. \tag{23}$$

We select $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$, $a(H) > 1$. Thus, in view of $\tilde{\boldsymbol{x}}(t) \in \mathcal{I}$, we get

$$\widehat{U}(\cdot) = \left(\hat{\boldsymbol{u}}^1(\cdot), \ldots, \hat{\boldsymbol{u}}^p(\cdot) \right) \in \mathcal{A}^0(\hat{\boldsymbol{x}}), \tag{24}$$

where $\hat{\boldsymbol{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$ with $\hat{x}_\ell = x_\ell / a(H)$, $\ell = 1, \ldots, n$. Furthermore, from the definition of $a(H)$, we have that for $\ell = 1, \ldots, m$,

$$\hat{x}_\ell(t) = \frac{\tilde{x}_\ell(t)}{a(H)} \leqslant \frac{H_\ell}{a(H)} \leqslant \frac{H_\ell}{H_\ell/(H_\ell - 2\varepsilon^\delta)} = H_\ell - 2\varepsilon^\delta, \tag{25}$$

and for $j = m+1, \ldots, n$,

$$\hat{x}_\ell(t) = \frac{\tilde{x}_\ell(t)}{a(H)} + \left(\frac{1}{a(H)} - 1 \right) z_j t \leqslant \frac{H_\ell}{a(H)} \leqslant \frac{H_\ell}{H_\ell/(H_\ell - 2\varepsilon^\delta)} = H_\ell - 2\varepsilon^\delta. \tag{26}$$

On the other hand, it follows from (22) and (23) that

$$\left| \hat{x}_j(t) - \tilde{x}_j(t) \right| = \left| \frac{\tilde{x}_j(t)}{a(H)} - \tilde{x}_j(t) \right| \leqslant \frac{2\varepsilon^\delta}{\min_{1 \leqslant i \leqslant n}\{H_i\}} \cdot \tilde{x}_j(t), \quad j = 1, \ldots, m, \tag{27}$$

and

$$\begin{aligned} \left| \hat{x}_j(t) - \tilde{x}_j(t) \right| &= \left| \frac{\tilde{x}_j(t)}{a(H)} + \left(\frac{1}{a(H)} - 1 \right) z_j t - \tilde{x}_j(t) \right| \\ &\leqslant \frac{2\varepsilon^\delta}{\min_{1 \leqslant i \leqslant n}\{H_i\}} \cdot \left(\left| \tilde{x}_j(t) \right| + z_j t \right), \quad j = m+1, \ldots, n. \end{aligned} \tag{28}$$

Hence,

$$\left| \hat{\boldsymbol{x}}(t) - \tilde{\boldsymbol{x}}(t) \right| \leqslant \frac{2\varepsilon^\delta}{\min_{1 \leqslant j \leqslant n}\{H_j\}} \cdot \left(\left| \tilde{\boldsymbol{x}}(t) \right| + |\boldsymbol{z}| t \right). \tag{29}$$

From the definition of $\tilde{\boldsymbol{u}}(\cdot)$, we have that for $i = 1, \ldots, p$,

$$\left| \tilde{\boldsymbol{u}}^i(t) - \hat{\boldsymbol{u}}^i(t) \right| = \left| 1 - \frac{1}{a(H)} \right| \cdot \left| \tilde{\boldsymbol{u}}^i(t) \right| \leqslant \frac{2\varepsilon^\delta}{\min_{1 \leqslant j \leqslant n}\{H_j\}} \cdot \left| \tilde{\boldsymbol{u}}^i(t) \right|. \tag{30}$$

Based on the difference between $\hat{\boldsymbol{x}}(t)$ and $\tilde{\boldsymbol{x}}(t)$ given in (29) and the difference between $\widehat{U}(t)$ and $\widetilde{U}(t)$ given in (30), we next estimate the difference between $J(\boldsymbol{x}, \widetilde{U}(\cdot))$ and $J(\hat{\boldsymbol{x}}, \widehat{U}(\cdot))$. First we have

$$\begin{aligned}
&\left|J\big(\boldsymbol{x},\widetilde{U}(\cdot)\big)-J\big(\hat{\boldsymbol{x}},\widehat{U}(\cdot)\big)\right|\\
&\quad=\left|\int_0^\infty e^{-\rho t}\left[h\big(\tilde{\boldsymbol{x}}(t)\big)+\sum_{i=1}^p\gamma_i c\big(\tilde{\boldsymbol{u}}^i(t)\big)\right]\mathrm{d}t-\int_0^\infty e^{-\rho t}\left[h\big(\hat{\boldsymbol{x}}(t)\big)+\sum_{i=1}^p\gamma_i c\big(\hat{\boldsymbol{u}}^i(t)\big)\right]\mathrm{d}t\right|\\
&\quad\leqslant\int_0^\infty e^{-\rho t}\left|h\big(\tilde{\boldsymbol{x}}(t)\big)-h\big(\hat{\boldsymbol{x}}(t)\big)\right|\mathrm{d}t+\int_0^\infty e^{-\rho t}\left|\sum_{i=1}^p\gamma_i c\big(\tilde{\boldsymbol{u}}^i(t)\big)-\sum_{i=1}^p\gamma_i c\big(\hat{\boldsymbol{u}}^i(t)\big)\right|\mathrm{d}t. \qquad (31)
\end{aligned}$$

Assumption (A2) and (29) imply that

$$\begin{aligned}
&\int_0^\infty e^{-\rho t}\left|h\big(\tilde{\boldsymbol{x}}(t)\big)-h\big(\hat{\boldsymbol{x}}(t)\big)\right|\mathrm{d}t\\
&\quad\leqslant C_0\int_0^\infty e^{-\rho t}\left[1+\left|\hat{\boldsymbol{x}}(t)\right|^{K_h}+\left|\tilde{\boldsymbol{x}}(t)\right|^{K_h}\right]\cdot\left|\hat{\boldsymbol{x}}(t)-\tilde{\boldsymbol{x}}(t)\right|\mathrm{d}t\\
&\quad\leqslant\frac{2C_0\varepsilon^\delta}{\min_{1\leqslant j\leqslant n}\{H_j\}}\int_0^\infty e^{-\rho t}\left[1+\left|\hat{\boldsymbol{x}}(t)\right|^{K_h}+\left|\tilde{\boldsymbol{x}}(t)\right|^{K_h}\right]\cdot\left(\left|\tilde{\boldsymbol{x}}(t)\right|+|zt|\right)\mathrm{d}t. \qquad (32)
\end{aligned}$$

Note that

$$\left|\hat{\boldsymbol{x}}(t)\right|\leqslant C_1\big(|\boldsymbol{x}|+t\big),\quad \left|\tilde{\boldsymbol{x}}(t)\right|\leqslant C_1\big(|\boldsymbol{x}|+t\big),\quad\text{and}\quad\int_0^\infty e^{-\rho t}t^{K_h+1}\,\mathrm{d}t\leqslant C_1,$$

for some $C_1>0$. Therefore, (32) implies that

$$\int_0^\infty e^{-\rho t}\left|h\big(\tilde{\boldsymbol{x}}(t)\big)-h\big(\hat{\boldsymbol{x}}(t)\big)\right|\mathrm{d}t\leqslant C_2\big(1+|\boldsymbol{x}|^{K_h+1}\big)\varepsilon^\delta, \qquad (33)$$

for some $C_2>0$. In the same way, Assumption (A2) and (30) imply that

$$\int_0^\infty\sum_{i=1}^p\gamma_i e^{-\rho t}\left|c\big(\hat{\boldsymbol{u}}^i(t)\big)-c\big(\tilde{\boldsymbol{u}}^i(t)\big)\right|\mathrm{d}t\leqslant C_3\varepsilon^\delta, \qquad (34)$$

for some $C_3>0$ and $\varepsilon<1$. Combining (31) and (33)–(34), we get

$$\left|J\big(\boldsymbol{x},\widetilde{U}(\cdot)\big)-J\big(\hat{\boldsymbol{x}},\widehat{U}(\cdot)\big)\right|\leqslant C_4\big(1+|\boldsymbol{x}|^{K_h+1}\big)\varepsilon^\delta, \qquad (35)$$

for some $C_4>0$. Consequently, (20) gives that for $\varepsilon\in(0,\varepsilon_1\wedge 1)$,

$$V(\boldsymbol{x})+\left[C_4\big(1+|\boldsymbol{x}|^{K_h+1}\big)+1\right]\varepsilon^\delta\geqslant\int_0^\infty e^{-\rho t}\left[h\big(\hat{\boldsymbol{x}}(t)\big)+\sum_{i=1}^p\gamma_i c\big(\hat{\boldsymbol{u}}^i(t)\big)\right]\mathrm{d}t. \qquad (36)$$

Finally, we let

$$\begin{aligned}
&\boldsymbol{x}(\delta)=\hat{\boldsymbol{x}}+(1,\dots,1)'\varepsilon^\delta,\\
&\boldsymbol{u}^i(\delta,\varepsilon,t)=\hat{\boldsymbol{u}}^i(t),\quad i=1,\dots,p,\\
&U(\delta,\varepsilon,\cdot)=\big(\boldsymbol{u}^1(\delta,\varepsilon,\cdot),\dots,\boldsymbol{u}^p(\delta,\varepsilon,\cdot)\big),
\end{aligned}$$

and $\boldsymbol{x}(\delta,\varepsilon,t)$ be the trajectory under $U(\delta,\varepsilon,t)$ with $\boldsymbol{x}(\delta,\varepsilon,0)=\boldsymbol{x}(\delta)$. Then, (25) and (26) imply that for $j=1,\dots,m$,

$$x_j(\delta,\varepsilon,t)=\varepsilon^\delta+\hat{x}_j(t)\leqslant H_j-\varepsilon^\delta,$$

and for $j=m+1,\dots,n$

$$x_j(\delta,\varepsilon,t)=\varepsilon^\delta+\hat{x}_j(t)\leqslant H_j-\varepsilon^\delta.$$

Clearly, for $j=1,\dots,m$,

$$x_j(\delta,\varepsilon,t)=\varepsilon^\delta+\hat{x}_j(t)\geqslant\varepsilon^\delta.$$

Thus,

$$U(\delta,\varepsilon,\cdot)\in\mathcal{A}^0\big(\boldsymbol{x}(\delta)\big),$$

and (15)–(17) hold. Note that

$$\big|\boldsymbol{x}(\delta,\varepsilon,t)-\hat{\boldsymbol{x}}(t)\big|\leqslant\varepsilon^\delta.$$

Similar to (35), there exists a positive constant C_5 such that

$$\big|J\big(\boldsymbol{x}(\delta),U(\delta,\varepsilon,\cdot)\big)-J\big(\hat{\boldsymbol{x}},\widehat{U}(\cdot)\big)\big|\leqslant C_5\big(1+|\boldsymbol{x}|^{K_h+1}\big)\varepsilon^\delta. \tag{37}$$

Consequently, (18) follows from (36) and (37). □

With Lemmas 4.1–4.3 in hand, we can derive our main result.

Theorem 4.1. *Let Assumptions* (A.1) *and* (A.2) *hold. Then for any* $\delta\in(0,\frac{1}{2})$, *there exist constants* $C>0$ *and* $\varepsilon_0>0$ *such that for* $\varepsilon\in(0,\varepsilon_0)$,

$$\big|V^\varepsilon(\boldsymbol{x},\boldsymbol{k})-V(\boldsymbol{x})\big|\leqslant C\big(1+|\boldsymbol{x}|^{K_h+1}\big)\varepsilon^\delta. \tag{38}$$

This implies in particular that $\lim_{\varepsilon\to0}V^\varepsilon(\boldsymbol{x},\boldsymbol{k})=V(\boldsymbol{x})$.

Remark 4.1. Theorem 4.1 says that the problem $\mathcal{P}^0$ is indeed a limiting problem in the sense that the $V^\varepsilon(\boldsymbol{x},\boldsymbol{k})$ of $\mathcal{P}^\varepsilon$ converges to $V(\boldsymbol{x})$ of $\mathcal{P}^0$. Moreover, it gives the corresponding convergence rate.

Proof. We outline the major steps in the proof. First we prove $V^\varepsilon(\boldsymbol{x},\boldsymbol{k})<V(\boldsymbol{x})+C(1+|\boldsymbol{x}|^{K_h})\varepsilon^\delta$ by constructing an admissible control $U(\varepsilon,t)\in\mathcal{A}^\varepsilon(\boldsymbol{x},\boldsymbol{k})$ of $\mathcal{P}^\varepsilon$ from the near-optimal control of the limiting problem $\mathcal{P}^0$, and by estimating the difference between the state trajectories corresponding to these two controls. Then we establish the opposite inequality, namely, $V^\varepsilon(\boldsymbol{x},\boldsymbol{k})>V(\boldsymbol{x})-C(1+|\boldsymbol{x}|^{K_h})\varepsilon^\delta$, by constructing a control of the limiting problem $\mathcal{P}^0$ from a near-optimal control of $\mathcal{P}^\varepsilon$ and using Assumption (A.2).

In order to show that

$$V^\varepsilon(\boldsymbol{x},\boldsymbol{k})-V(\boldsymbol{x})\leqslant C\big(1+|\boldsymbol{x}|^{K_h}\big)\varepsilon^\delta, \tag{39}$$

we can choose, in view of Lemma 4.3, $\boldsymbol{x}(\delta)\in\mathcal{I}$ and

$$U(\delta,\varepsilon,\cdot)=\big(\boldsymbol{u}^1(\delta,\varepsilon,\cdot),\dots,\boldsymbol{u}^p(\delta,\varepsilon,\cdot)\big)\in\mathcal{A}^0\big(\boldsymbol{x}(\delta)\big),$$

such that

$$\big|\boldsymbol{x}(\delta)-\boldsymbol{x}\big| \leqslant C_0\big(1+|\boldsymbol{x}|\big)\varepsilon^{\delta}, \tag{40}$$

$$\inf_{0\leqslant s<\infty} x_j(\delta,\varepsilon,s) \geqslant \varepsilon^{\delta}, \quad j=1,\dots,m, \tag{41}$$

$$\sup_{0\leqslant s<\infty} x_j(\delta,\varepsilon,s) \leqslant H_j-\varepsilon^{\delta}, \quad j=1,\dots,n, \tag{42}$$

and

$$\begin{aligned} V(\boldsymbol{x})+C_0\big(1+|\boldsymbol{x}|^{K_h+1}\big)\varepsilon^{\delta} &\geqslant J\big(\boldsymbol{x}(\delta),U(\delta,\varepsilon,\cdot)\big) \\ &= \int_0^\infty e^{-\rho t}\Bigg[h\big(\boldsymbol{x}(\delta,\varepsilon,t)\big)+\sum_{i=1}^p \gamma_i c\big(\boldsymbol{u}^i(\delta,\varepsilon,t)\big)\Bigg]\,\mathrm{d}t, \end{aligned} \tag{43}$$

where $\boldsymbol{x}(\delta,\varepsilon,t)$ is the state trajectory under the control $U(\delta,\varepsilon,t)$ with $\boldsymbol{x}(\delta,\varepsilon,0)=\boldsymbol{x}(\delta)$. Let

$$\hat{\boldsymbol{u}}^i(\varepsilon,t)=I_{\{\boldsymbol{k}(\varepsilon,t)=\boldsymbol{k}^i\}}\boldsymbol{u}^i(\delta,\varepsilon,t), \qquad \hat{\boldsymbol{u}}(\varepsilon,t)=\sum_{i=1}^p I_{\{\boldsymbol{k}(\varepsilon,t)=\boldsymbol{k}^i\}}\boldsymbol{u}^i(\delta,\varepsilon,t),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{x}}(\varepsilon,t)=\mathcal{L}\big(\hat{\boldsymbol{u}}(\varepsilon,t),\boldsymbol{z}\big), \qquad \hat{\boldsymbol{x}}(\varepsilon,0)=\boldsymbol{x}(\delta).$$

Generally, the control $\hat{\boldsymbol{u}}(\varepsilon,t)$ may not be admissible. We need to make it admissible and still satisfy (39). This modification will be done in two steps. First we modify $\hat{\boldsymbol{u}}(\varepsilon,t)$ such that the works-in-process of its state trajectory are nonnegative. That this can be done is asserted in the following lemma. Its proof is given in Appendix A.

Lemma 4.4. *There are $\bar{\boldsymbol{u}}(\varepsilon,t)$ and constants $C_1>0$, $\varepsilon_1>0$ and $\kappa_1>0$ such that for any $\varepsilon\in(0,\varepsilon_1)$, $(i,j)\in A$ and $i\neq m+1,\dots,n$,*

$$\begin{aligned} &\bar{u}_{i,j}(\varepsilon,t)\leqslant \hat{u}_{i,j}(\varepsilon,t), \\ &\bar{x}_\ell(\varepsilon,t)\geqslant 0, \quad \ell=1,\dots,m, \end{aligned} \tag{44}$$

$$\begin{aligned} &\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{m+1}}\right\}\cdot\big|\bar{u}_{i,j}(\varepsilon,t)-\hat{u}_{i,j}(\varepsilon,t)\big|\,\mathrm{d}t \leqslant C_1\exp\big\{-\kappa_1\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\big\}, \\ &\quad (i,j)\in A,\ i\neq m+1,\dots,n, \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{m+1}}\right\}\cdot\Pr\left(\bar{x}_\ell(\varepsilon,t)\geqslant H_\ell-\left(\frac{1}{2}-\sum_{j=1}^{\ell}\frac{1}{2^{j+1}}\right)\varepsilon^{\delta}\right)\mathrm{d}t \\ &\quad \leqslant C_1\exp\big\{-\kappa_1\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\big\}, \quad \ell=1,\dots,n, \end{aligned} \tag{46}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\boldsymbol{x}}(\varepsilon,t)=\mathcal{L}\big(\bar{\boldsymbol{u}}(\varepsilon,t),\boldsymbol{z}\big), \qquad \bar{\boldsymbol{x}}(\varepsilon,0)=\boldsymbol{x}(\delta).$$

Because $\bar{\boldsymbol{u}}(\varepsilon, t)$ is not admissible, it may violate the upper bound. So we need to modify it further. This modification provides a feasible control as stated in the following lemma. We relegate its proof to Appendix A.

Lemma 4.5. *There exist a control $\boldsymbol{u}(\varepsilon, t) \in \mathcal{A}^{\varepsilon}(\boldsymbol{x}(\delta), \boldsymbol{k})$ and constants $C_2 > 0$, $\varepsilon_2 > 0$, and $\kappa_2 > 0$, such that for any $\varepsilon \in (0, \varepsilon_2)$, $(i, j) \in A$, and $i \neq m+1, \ldots, n$,*

$$u_{i,j}(\varepsilon, t) \leqslant \bar{u}_{i,j}(\varepsilon, t)$$

and

$$\mathsf{E} \int_0^\infty e^{-\rho t/2} \left| u_{i,j}(\varepsilon, t) - \bar{u}_{i,j}(\varepsilon, t) \right| \mathrm{d}t \leqslant C_2 \exp\left\{ -\kappa_2 \varepsilon^{-\frac{1}{4}(\frac{1}{2} - \delta)} \right\},$$
$$(i, j) \in A, \quad i \neq m+1, \ldots, n. \tag{47}$$

Let

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{x}(\varepsilon, t) = \mathcal{L}\big(\boldsymbol{u}(\varepsilon, t), \boldsymbol{z}\big), \qquad \boldsymbol{x}(\varepsilon, 0) = \boldsymbol{x}(\delta).$$

Based on Lemmas 4.4 and 4.5, we can estimate

$$\begin{aligned} & J^{\varepsilon}\big(\boldsymbol{x}(\delta), \boldsymbol{k}, \boldsymbol{u}(\varepsilon, \cdot)\big) - J\big(\boldsymbol{x}(\delta), U(\delta, \varepsilon, \cdot)\big) \\ &\quad = \mathsf{E} \int_0^\infty e^{-\rho t} \big[h\big(\boldsymbol{x}(\varepsilon, t)\big) - h\big(\bar{\boldsymbol{x}}(\varepsilon, t)\big) \big] \, \mathrm{d}t + \mathsf{E} \int_0^\infty e^{-\rho t} \big[c\big(\boldsymbol{u}(\varepsilon, t)\big) - c\big(\bar{\boldsymbol{u}}(\varepsilon, t)\big) \big] \, \mathrm{d}t \\ &\qquad + \mathsf{E} \int_0^\infty e^{-\rho t} \big[h\big(\bar{\boldsymbol{x}}(\varepsilon, t)\big) - h\big(\hat{\boldsymbol{x}}(\varepsilon, t)\big) \big] \, \mathrm{d}t + \mathsf{E} \int_0^\infty e^{-\rho t} \big[c\big(\bar{\boldsymbol{u}}(\varepsilon, t)\big) - c\big(\hat{\boldsymbol{u}}(\varepsilon, t)\big) \big] \, \mathrm{d}t \\ &\qquad + \mathsf{E} \int_0^\infty e^{-\rho t} \big[h\big(\hat{\boldsymbol{x}}(\varepsilon, t)\big) - h\big(\boldsymbol{x}(\delta, \varepsilon, t)\big) \big] \, \mathrm{d}t \\ &\qquad + \mathsf{E} \int_0^\infty e^{-\rho t} \left[c\big(\hat{\boldsymbol{u}}(\varepsilon, t)\big) - \sum_{\ell=1}^{p} \gamma_\ell c\big(\boldsymbol{u}^\ell(\delta, \varepsilon, t)\big) \right] \mathrm{d}t. \end{aligned} \tag{48}$$

By Assumption (A2), there is a constant $C_3 > 0$ such that

$$\begin{aligned} & \mathsf{E} \int_0^\infty e^{-\rho t} \big[h\big(\boldsymbol{x}(\varepsilon, t)\big) - h\big(\bar{\boldsymbol{x}}(\varepsilon, t)\big) \big] \, \mathrm{d}t \\ &\qquad \leqslant C_3 \big(1 + \left|\boldsymbol{x}(\delta)\right|^{K_h}\big) \mathsf{E} \int_0^\infty e^{-\rho t} \big(1 + t^{K_h}\big) \cdot \left| \boldsymbol{x}(\varepsilon, t) - \bar{\boldsymbol{x}}(\varepsilon, t) \right| \mathrm{d}t. \end{aligned} \tag{49}$$

Note that $\int_s^\infty e^{-\rho t/2}(1 + t^{K_h}) \, \mathrm{d}t$ is bounded on $[0, \infty)$. Hence, there exist constants $C_4 > 0$ and $\varepsilon_{11} > 0$ such that for $\varepsilon \in (0, \varepsilon_{11})$,

$$
\begin{aligned}
&\mathsf{E}\int_0^\infty e^{-\rho t}\big[h\big(\boldsymbol{x}(\varepsilon,t)\big)-h\big(\bar{\boldsymbol{x}}(\varepsilon,t)\big)\big]\,\mathrm{d}t\\
&\quad\leqslant C_3\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\mathsf{E}\int_0^\infty e^{-\rho t}\big(1+t^{K_h}\big)\sum_{(i,j)\in A}\int_0^t\big|u_{i,j}(\varepsilon,s)-\bar{u}_{i,j}(\varepsilon,s)\big|\,\mathrm{d}s\,\mathrm{d}t\\
&\quad=C_3\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\mathsf{E}\sum_{(i,j)\in A}\int_0^\infty\left[\big|u_{i,j}(\varepsilon,s)-\bar{u}_{i,j}(\varepsilon,s)\big|\int_s^\infty e^{-\rho t}\big(1+t^{K_h}\big)\,\mathrm{d}t\right]\mathrm{d}s\\
&\quad\leqslant C_3\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\\
&\qquad\times\mathsf{E}\sum_{(i,j)\in A}\int_0^\infty e^{-\rho s/2}\left[\big|u_{i,j}(\varepsilon,s)-\bar{u}_{i,j}(\varepsilon,s)\big|\int_s^\infty e^{-\rho t/2}\big(1+t^{K_h}\big)\,\mathrm{d}t\right]\mathrm{d}s\\
&\quad\leqslant C_3\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\cdot\int_0^\infty e^{-\rho t/2}\big(1+t^{K_h}\big)\,\mathrm{d}t\\
&\qquad\times\mathsf{E}\sum_{(i,j)\in A}\int_0^\infty e^{-\rho s/2}\big[\big|u_{i,j}(\varepsilon,s)-\bar{u}_{i,j}(\varepsilon,s)\big|\big]\,\mathrm{d}s\\
&\quad\leqslant C_4\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\varepsilon^{\delta}\quad\big(\text{by (47)}\big).
\end{aligned}\tag{50}
$$

In the same way, we get

$$
\mathsf{E}\int_0^\infty e^{-\rho t}\big[h\big(\bar{\boldsymbol{x}}(\varepsilon,t)\big)-h\big(\hat{\boldsymbol{x}}(\varepsilon,t)\big)\big]\,\mathrm{d}t\leqslant C_5\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\varepsilon^{\delta},\tag{51}
$$

$$
\mathsf{E}\int_0^\infty e^{-\rho t}\big[c\big(\boldsymbol{u}(\varepsilon,t)\big)-c\big(\bar{\boldsymbol{u}}(\varepsilon,t)\big)\big]\,\mathrm{d}t\leqslant C_5\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\varepsilon^{\delta},\tag{52}
$$

and

$$
\mathsf{E}\int_0^\infty e^{-\rho t}\big[c\big(\bar{\boldsymbol{u}}(\varepsilon,t)\big)-c\big(\hat{\boldsymbol{u}}(\varepsilon,t)\big)\big]\,\mathrm{d}t\leqslant C_5\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\varepsilon^{\delta},\tag{53}
$$

for some $C_5>0$ and $\varepsilon\in(0,\varepsilon_{11})$. Similar to (50), by the definition of $\hat{\boldsymbol{x}}(\varepsilon,t)$, there is a positive constant $C_6>0$ such that

$$
\begin{aligned}
&\mathsf{E}\int_0^\infty e^{-\rho t}\big[h\big(\hat{\boldsymbol{x}}(\varepsilon,t)\big)-h\big(\boldsymbol{x}(\delta,\varepsilon,t)\big)\big]\,\mathrm{d}t\\
&\quad\leqslant C_6\big(1+\big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\sum_{\ell=1}^{p}\mathsf{E}\left|\int_0^\infty e^{-\rho s/2}\big[I_{\{\boldsymbol{k}(\varepsilon,s)=\boldsymbol{k}^\ell\}}-\gamma_\ell\big]\boldsymbol{u}^\ell(\delta,\varepsilon,s)\,\mathrm{d}s\right|
\end{aligned}
$$

$$= C_6\big(1 + \big|\boldsymbol{x}(\delta)\big|^{K_h}\big) \sum_{\ell=1}^{p} \mathsf{E}\left| \int_0^{-\ln\varepsilon/\rho} e^{-\rho s/2}\big[I_{\{\boldsymbol{k}(\varepsilon,s)=\boldsymbol{k}^\ell\}} - \gamma_\ell\big]\boldsymbol{u}^\ell(\delta,\varepsilon,s)\,\mathrm{d}s\right|$$

$$+ C_6\big(1 + \big|\boldsymbol{x}(\delta)\big|^{K_h}\big) \sum_{\ell=1}^{p} \mathsf{E}\left| \int_{-\ln\varepsilon/\rho}^{\infty} e^{-\rho s/2}\big[I_{\{\boldsymbol{k}(\varepsilon,s)=\boldsymbol{k}^\ell\}} - \gamma_\ell\big]\boldsymbol{u}^\ell(\delta,\varepsilon,s)\,\mathrm{d}s\right|. \tag{54}$$

It follows from the boundedness of $\boldsymbol{u}^\ell(\delta,\varepsilon,t)$, (54), and Lemma 4.2 that there exist positive constants C_7 and ε_{12} ($< \varepsilon_{11}$) such that for $\varepsilon \in (0, \varepsilon_{12})$,

$$\mathsf{E}\int_0^\infty e^{-\rho t}\big[h\big(\hat{\boldsymbol{x}}(\varepsilon,t)\big) - h\big(\boldsymbol{x}(\delta,\varepsilon,t)\big)\big]\,\mathrm{d}t \leqslant C_7\big(1 + \big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\varepsilon^\delta. \tag{55}$$

Furthermore, from Lemma 4.1 and the boundedness of $c(\boldsymbol{u}^\ell(\delta,\varepsilon,t))$,

$$\begin{aligned} &\mathsf{E}\int_0^\infty e^{-\rho t}\left[c\big(\hat{\boldsymbol{u}}(\varepsilon,t)\big) - \sum_{\ell=1}^{p}\gamma_\ell c\big(\boldsymbol{u}^\ell(\delta,\varepsilon,t)\big)\right]\mathrm{d}t \\ &\quad\leqslant \mathsf{E}\int_0^\infty e^{-\rho t}\sum_{\ell=1}^{p}\big(I_{\{\boldsymbol{k}(\varepsilon,t)=\boldsymbol{k}^\ell\}} - \gamma_\ell\big)c\big(\boldsymbol{u}^\ell(\delta,\varepsilon,t)\big)\,\mathrm{d}t \\ &\quad\leqslant C_8\varepsilon, \end{aligned} \tag{56}$$

for some $C_8 > 0$. Therefore, it follow from (48)–(56) that there are positive constants C_9 and ε_{13} ($< \varepsilon_{12} \wedge 1$) such that for $\varepsilon \in (0, \varepsilon_{13})$,

$$J^\varepsilon\big(\boldsymbol{x}(\delta), \boldsymbol{k}, \boldsymbol{u}(\varepsilon,\cdot)\big) - J\big(\boldsymbol{x}(\delta), U(\delta,\varepsilon,\cdot)\big) \leqslant C_9\big(1 + |\boldsymbol{x}|^{K_h}\big)\varepsilon^\delta. \tag{57}$$

On the other hand, from the Lipschitz continuity of $V^\varepsilon(\boldsymbol{x},\boldsymbol{k})$, (40) implies

$$\big|V^\varepsilon(\boldsymbol{x},\boldsymbol{k}) - V^\varepsilon\big(\boldsymbol{x}(\delta),\boldsymbol{k}\big)\big| \leqslant C_{10}\big(1 + |\boldsymbol{x}|^{K_h}\big)\varepsilon^\delta, \tag{58}$$

for some $C_{10} > 0$. In view of $J^\varepsilon(\boldsymbol{x}(\delta), \boldsymbol{k}, \boldsymbol{u}(\varepsilon,\cdot)) \geqslant V^\varepsilon(\boldsymbol{x}(\delta),\boldsymbol{k})$, (39) follows from (43) and (57)–(58).

We now show the opposite inequality, that is,

$$V^\varepsilon(\boldsymbol{x},\boldsymbol{k}) - V(\boldsymbol{x}) \geqslant C\big(1 + |\boldsymbol{x}|^{K_h}\big)\varepsilon^\delta. \tag{59}$$

Similar to Lemma 4.3, we can prove that there exist $\boldsymbol{x}(\delta) \in \mathcal{I}$ and a control $\boldsymbol{u}(\varepsilon,\cdot) \in \mathcal{A}^\varepsilon(\boldsymbol{x}(\delta),\boldsymbol{k})$ such that

$$\big|\boldsymbol{x}(\delta) - \boldsymbol{x}\big| \leqslant \widehat{C}_1\big(1 + |\boldsymbol{x}|\big)\varepsilon^\delta, \tag{60}$$

$$\min_{1\leqslant j\leqslant m}\ \inf_{0\leqslant t<\infty} \mathsf{E}\big[x_j(\varepsilon,t)\big] \geqslant \varepsilon^\delta, \tag{61}$$

$$\sup_{0\leqslant s<\infty} \mathsf{E}\big[x_j(\varepsilon,s)\big] \leqslant H_j - \varepsilon^\delta, \quad j = 1,\ldots,n, \tag{62}$$

and

$$\mathsf{E}\int_0^\infty e^{-\rho t}\big[h\big(\boldsymbol{x}(\varepsilon,t)\big) + c\big(\boldsymbol{u}(\varepsilon,t)\big)\big]\,\mathrm{d}t \leqslant V^\varepsilon(\boldsymbol{x},\boldsymbol{k}) + \widehat{C}_1\big(1 + |\boldsymbol{x}|^{K_h+1}\big)\varepsilon^\delta, \tag{63}$$

for some $\widehat{C}_1 > 0$, where $\boldsymbol{x}(\varepsilon, t)$ is the state trajectory under the control $\boldsymbol{u}(\varepsilon, t)$ with the initial condition $\boldsymbol{x}(\varepsilon, 0) = \boldsymbol{x}(\delta)$. Now we choose $\widetilde{U}(\varepsilon, \cdot) = (\tilde{\boldsymbol{u}}^1(\varepsilon, \cdot), \ldots, \tilde{\boldsymbol{u}}^p(\varepsilon, \cdot))$ defined by

$$\tilde{\boldsymbol{u}}^\ell(\varepsilon, t) = \mathsf{E}\big[\boldsymbol{u}(\varepsilon, t) \,\big|\, \boldsymbol{k}(\varepsilon, t) = \boldsymbol{k}^\ell\big], \quad \ell = 1, \ldots, p.$$

Then we have

$$\begin{cases} \mathsf{E}\big[x_k(\varepsilon, t)\big] = x_k(\delta) + \displaystyle\int_0^t \Bigg[\sum_{\ell=1}^p \Pr\big(\boldsymbol{k}(\varepsilon, t) = \boldsymbol{k}^\ell\big) \sum_{i=-n_0+1}^{k-1} \tilde{u}_{i,k}^\ell(\varepsilon, s) \\ \qquad\qquad - \displaystyle\sum_{\ell=1}^p \Pr\big(\boldsymbol{k}(\varepsilon, t) = \boldsymbol{k}^\ell\big) \sum_{i=k+1}^{n} \tilde{u}_{k,i}^\ell(\varepsilon, s)\Bigg]\,\mathsf{d}s, \\ \quad k = 1, \ldots, m, \\ \mathsf{E}\big[x_k(\varepsilon, t)\big] = x_k(\delta) + \displaystyle\int_0^t \Bigg[\sum_{\ell=1}^p \Pr\big(\boldsymbol{k}(\varepsilon, t) = \boldsymbol{k}^\ell\big) \sum_{i=-n_0+1}^{m} \tilde{u}_{i,k}^\ell(\varepsilon, s) - z_k\Bigg]\,\mathsf{d}s, \\ \quad k = m+1, \ldots, n. \end{cases}$$

Define

$$\begin{cases} \tilde{x}_k(\varepsilon, t) = x_k(\delta) + \displaystyle\int_0^t \Bigg[\sum_{j=1}^p \gamma_j \sum_{i=-n_0+1}^{k-1} \tilde{u}_{i,k}^j(\varepsilon, s) - \sum_{j=1}^p \gamma_j \sum_{i=k+1}^{n} \tilde{u}_{k,i}^j(\varepsilon, s)\Bigg]\,\mathsf{d}s, \\ \quad k = 1, \ldots, m, \\ \tilde{x}_k(\varepsilon, t) = x_k(\delta) + \displaystyle\int_0^t \Bigg[\sum_{j=1}^p \gamma_j \sum_{i=-n_0+1}^{m} \tilde{u}_{i,k}^j(\varepsilon, s) - z_k\Bigg]\,\mathsf{d}s, \quad k = m+1, \ldots, n. \end{cases}$$

Using Lemma 4.1, we have

$$\begin{aligned} \Bigg|\mathsf{E}\big[\boldsymbol{u}(\varepsilon, t)\big] - \sum_{\ell=1}^p \gamma_\ell \tilde{\boldsymbol{u}}^\ell(\varepsilon, t)\Bigg| &= \Bigg|\sum_{\ell=1}^p \big[\Pr\big(\boldsymbol{k}(\varepsilon, t) = \boldsymbol{k}^\ell\big) - \gamma_\ell\big]\tilde{\boldsymbol{u}}^\ell(\varepsilon, t)\Bigg| \\ &\leqslant \widehat{C}_2\big(\varepsilon + e^{-\kappa\varepsilon^{-1}t}\big), \end{aligned} \tag{64}$$

for some $\widehat{C}_2 > 0$. Then for $k = 1, \ldots, m$,

$$\begin{aligned} \big|\tilde{x}_k(\varepsilon, t) - \mathsf{E}\big[x_k(\varepsilon, t)\big]\big| &= \Bigg|\int_0^t \Bigg\{\sum_{\ell=1}^p \gamma_\ell \Bigg(\sum_{i=-n_0+1}^{k-1} \tilde{u}_{i,k}^\ell(\varepsilon, s) - \sum_{i=k+1}^{n} \tilde{u}_{k,i}^\ell(\varepsilon, s)\Bigg) \\ &\qquad - \mathsf{E}\Bigg[\sum_{i=-n_0+1}^{k-1} u_{i,k}(\varepsilon, s) - \sum_{i=k+1}^{n} u_{k,i}(\varepsilon, s)\Bigg]\Bigg\}\,\mathsf{d}s\Bigg| \\ &= \Bigg|\int_0^t \Bigg(\sum_{\ell=1}^p \gamma_\ell \sum_{i=-n_0+1}^{k-1} \tilde{u}_{i,k}^\ell(\varepsilon, s) - \mathsf{E}\sum_{i=-n_0+1}^{k-1} u_{i,k}(\varepsilon, s)\Bigg)\,\mathsf{d}s \\ &\qquad - \int_0^t \Bigg(\sum_{\ell=1}^p \gamma_\ell \sum_{i=k+1}^{n} \tilde{u}_{k,i}^\ell(\varepsilon, s) - \mathsf{E}\sum_{i=k+1}^{n} u_{k,i}(\varepsilon, s)\Bigg)\,\mathsf{d}s\Bigg| \\ &\leqslant \widehat{C}_3(1+t)\varepsilon, \end{aligned} \tag{65}$$

and for $k = m+1, \dots, n$,

$$\begin{aligned}\left|\tilde{x}_k(\varepsilon,t) - \mathsf{E}\big[x_k(\varepsilon,t)\big]\right| &\leqslant \left|\int_0^t \left[\sum_{\ell=1}^{p} \gamma_\ell \sum_{i=-n_0+1}^{m} \tilde{u}_{i,k}^{\ell}(\varepsilon,s) - \mathsf{E}\sum_{i=-n_0+1}^{m} u_{i,k}(\varepsilon,s)\right] \mathsf{d}s\right| \\ &\leqslant \widehat{C}_3(1+t)\varepsilon, \end{aligned} \tag{66}$$

for some $\widehat{C}_3 > 0$. According to Lemma 5.4 of Sethi, Zhang and Zhou [6], there exists $\tau_\varepsilon > 0$ such that

$$\widehat{C}_3(1+\tau_\varepsilon)\varepsilon \leqslant \varepsilon^{\delta} \tag{67}$$

and

$$\int_{\tau_\varepsilon}^{\infty} e^{-\rho t}\big(1 + t^{K_h+1}\big)\, \mathsf{d}t \leqslant \widehat{C}_4 \varepsilon^{\delta}, \tag{68}$$

for some $\widehat{C}_4 > 0$. Therefore, if one defines

$$\check{u}_{i,k}^{\ell}(\varepsilon,t) = \begin{cases} \tilde{u}_{i,k}^{\ell}(\varepsilon,t), & 0 \leqslant t \leqslant \tau_\varepsilon, \\ 0, & t > \tau_\varepsilon, \end{cases}$$

for $\ell = 1, \dots, p$, and $(i,k) \in A$ with $i \neq m+1, \dots, n$, and lets $\check{\boldsymbol{x}}(\varepsilon,t)$ be the state trajectory under the control $\check{U}(\varepsilon,t) = (\check{\boldsymbol{u}}^1(\varepsilon,t), \dots, \check{\boldsymbol{u}}^p(\varepsilon,t))$ with $\check{\boldsymbol{x}}(\varepsilon,0) = \boldsymbol{x}(\delta)$, then (61)–(62) and (65)–(67) imply

$$\check{U}(\varepsilon,t) = \big(\check{\boldsymbol{u}}^1(\varepsilon,t), \dots, \check{\boldsymbol{u}}^p(\varepsilon,t)\big) \in \mathcal{A}^0\big(\boldsymbol{x}(\delta)\big).$$

It follows from (68) and Assumption (A.2) that

$$\begin{aligned} &\left|\int_0^{\infty} e^{-\rho t}\left[h\big(\check{\boldsymbol{x}}(\varepsilon,t)\big) + \sum_{\ell=1}^{p} \gamma_\ell c\big(\check{\boldsymbol{u}}^{\ell}(\varepsilon,t)\big)\right] \mathsf{d}t \right.\\ &\qquad \left. - \int_0^{\infty} e^{-\rho t}\left[h\big(\tilde{\boldsymbol{x}}(\varepsilon,t)\big) + \sum_{\ell=1}^{p} \gamma_\ell c\big(\tilde{\boldsymbol{u}}^{\ell}(\varepsilon,t)\big)\right] \mathsf{d}t\right| \\ &\quad = \left|\int_{\tau_\varepsilon}^{\infty} e^{-\rho t}\left[h\big(\check{\boldsymbol{x}}(\varepsilon,t)\big) + \sum_{\ell=1}^{p} \gamma_\ell c\big(\check{\boldsymbol{u}}^{\ell}(\varepsilon,t)\big)\right] \mathsf{d}t \right.\\ &\qquad \left. - \int_{\tau_\varepsilon}^{\infty} e^{-\rho t}\left[h\big(\tilde{\boldsymbol{x}}(\varepsilon,t)\big) + \sum_{\ell=1}^{p} \gamma_\ell c\big(\tilde{\boldsymbol{u}}^{\ell}(\varepsilon,t)\big)\right] \mathsf{d}t\right| \\ &\quad \leqslant \widehat{C}_5\big(1 + \left|\boldsymbol{x}(\delta)\right|^{K_h}\big)\varepsilon^{\delta}, \end{aligned} \tag{69}$$

for some $\widehat{C}_5 > 0$. In view of the convexity and the local Lipschitz continuity of $h(\cdot)$, Jensen's inequality and (65)–(66) yield

$$\begin{aligned} \mathsf{E}\big[h\big(\boldsymbol{x}(\varepsilon,t)\big)\big] &\geqslant h\big(\mathsf{E}\big[\boldsymbol{x}(\varepsilon,t)\big]\big) \\ &= h\big(\tilde{\boldsymbol{x}}(\varepsilon,t)\big) + \big[h\big(\mathsf{E}\big[\boldsymbol{x}(\varepsilon,t)\big]\big) - h\big(\tilde{\boldsymbol{x}}(\varepsilon,t)\big)\big] \end{aligned}$$

$$\geqslant h\big(\tilde{\boldsymbol{x}}(\varepsilon,t)\big) - C_0\big(1 + \big|\mathsf{E}\big[\boldsymbol{x}(\varepsilon,t)\big]\big|^{K_h} + \big|\tilde{\boldsymbol{x}}(\varepsilon,t)\big|^{K_h}\big)\big|\mathsf{E}\big[\boldsymbol{x}(\varepsilon,t)\big] - \tilde{\boldsymbol{x}}(\varepsilon,t)\big|$$
$$\geqslant h\big(\tilde{\boldsymbol{x}}(\varepsilon,t)\big) - \widehat{C}_6\varepsilon\big(1 + \big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\big(1 + t^{K_h+1}\big)(1+t), \tag{70}$$

for some $\widehat{C}_6 > 0$. In the same way, using Lemma 4.1, we can establish

$$\begin{aligned}\mathsf{E}\big[c\big(\boldsymbol{u}(\varepsilon,t)\big)\big] &= \sum_{\ell=1}^{p} \Pr\big(\boldsymbol{k}(\varepsilon,t) = \boldsymbol{k}^\ell\big)\mathsf{E}\big[c\big(\boldsymbol{u}(\varepsilon,t)\big) \;\big|\; \boldsymbol{k}(\varepsilon,t) = \boldsymbol{k}^\ell\big] \\ &\geqslant \sum_{\ell=1}^{p} \Pr\big(\boldsymbol{k}(\varepsilon,t) = \boldsymbol{k}^\ell\big)c\big(\tilde{\boldsymbol{u}}^\ell(\varepsilon,t)\big) \\ &\geqslant \sum_{\ell=1}^{p} \gamma_\ell c\big(\tilde{\boldsymbol{u}}^\ell(\varepsilon,t)\big) - \widehat{C}_7\big(\varepsilon + e^{-\kappa t/\varepsilon}\big),\end{aligned} \tag{71}$$

for some positive $\widehat{C}_7$. By combining (70) and (71), we obtain

$$\begin{aligned}&\mathsf{E}\int_0^\infty e^{-\rho t}\big[h\big(\boldsymbol{x}(\varepsilon,t)\big) + c\big(\boldsymbol{u}(\varepsilon,t)\big)\big]\,\mathsf{d}t \\ &\quad\geqslant \int_0^\infty e^{-\rho t}\Bigg[h\big(\tilde{\boldsymbol{x}}(\varepsilon,t)\big) + \sum_{j=1}^{p}\gamma_j c\big(\tilde{\boldsymbol{u}}^j(\varepsilon,t)\big)\Bigg]\,\mathsf{d}t - \widehat{C}_8\varepsilon,\end{aligned}$$

for some positive constant $\widehat{C}_8$. Thus, in view of $V(\boldsymbol{x}(\delta)) \leqslant J(\boldsymbol{x}(\delta), \check{\boldsymbol{u}}(\varepsilon,\cdot))$, (69) gives that there is a positive constant $\widehat{C}_9$ such that

$$\mathsf{E}\int_0^\infty e^{-\rho t}\big[h\big(\boldsymbol{x}(\varepsilon,t)\big) + c\big(\boldsymbol{u}(\varepsilon,t)\big)\big]\,\mathsf{d}t \geqslant V\big(\boldsymbol{x}(\delta)\big) - \widehat{C}_9\big(1 + \big|\boldsymbol{x}(\delta)\big|^{K_h}\big)\varepsilon^\delta. \tag{72}$$

On the other hand, the Lipschitz continuity of $V(\boldsymbol{x})$ and (60) imply

$$\big|V(\boldsymbol{x}) - V\big(\boldsymbol{x}(\delta)\big)\big| \leqslant \widehat{C}_{10}\big(1 + |\boldsymbol{x}|^{K_h}\big)\varepsilon^\delta,$$

for some $\widehat{C}_{10} > 0$. Consequently, the inequality (72) implies (59). □

5. Asymptotic optimal control

In this section, based on the proof of Theorem 4.1, we supply a procedure to construct an asymptotic optimal control.

Construction of an asymptotic optimal control

Step I. Pick an ε-optimal control $U(\varepsilon,\cdot) = (\boldsymbol{u}^1(\varepsilon,\cdot), \ldots, \boldsymbol{u}^p(\varepsilon,\cdot)) \in \mathcal{A}^0(\boldsymbol{x})$ for $\mathcal{P}^0$, i.e.,

$$\int_0^\infty e^{-\rho t}\Bigg[h\big(\boldsymbol{x}(\varepsilon,t)\big) + \sum_{j=1}^{p}\gamma_j c\big(\boldsymbol{u}^j(\varepsilon,t)\big)\Bigg]\,\mathsf{d}t < V(\boldsymbol{x}) + \varepsilon,$$

where $\boldsymbol{x}(\varepsilon,t)$ is the state trajectory under the control $U(\varepsilon,t)$ with $\boldsymbol{x}(\varepsilon,0)=\boldsymbol{x}$. Let

$$a(H)=\max_{1\leqslant j\leqslant n}\left\{\frac{H_j}{H_j-2\varepsilon^\delta}\right\}.$$

Define

$$\breve{u}_{i,j}(\varepsilon,t)=\frac{u_{i,j}(\varepsilon,t)}{a(H)},\quad (i,j)\in A,\ i\neq m+1,\dots,n.$$

Step II. Set

$$\hat{\boldsymbol{u}}(\varepsilon,t)=\sum_{\ell=1}^{p} I_{\{\boldsymbol{k}(\varepsilon,t)=\boldsymbol{k}^\ell\}}\breve{\boldsymbol{u}}^\ell(\varepsilon,t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{x}}(\varepsilon,t)=\mathcal{L}\big(\hat{\boldsymbol{u}}(\varepsilon,t),\boldsymbol{z}\big),\quad \hat{\boldsymbol{x}}(\varepsilon,0)=\frac{\boldsymbol{x}}{a(H)}+(1,\dots,1)'\varepsilon^\delta.$$

Set

$$\bar{u}_{i,j}(\varepsilon,t)=\hat{u}_{i,j}(\varepsilon,t),\quad i=-n_0+1,\dots,0,\ j=1,\dots,n.$$

Sub-step k ($k=1,\dots,m$): Set

$$B_k^\varepsilon=\Big\{t\colon \hat{x}_k^{k-1}(t)-\inf_{0\leqslant s\leqslant t}\hat{x}_k^{k-1}(s)=0 \text{ and } \hat{x}_k^{k-1}(t)<0\Big\},$$

where

$$\hat{x}_k^{k-1}(t)=\frac{x_k}{a(H)}+\varepsilon^\delta+\int_0^t\left[\sum_{i=-n_0+1}^{k-1}\bar{u}_{i,k}(\varepsilon,s)-\sum_{i=k+1}^{n}\hat{u}_{k,i}(\varepsilon,s)\right]\mathrm{d}s,$$

and $\hat{x}_k^0(t)=\hat{x}_k(t)$. For $t\in B_k^\varepsilon$ and $i=k+1,\dots,n$, choose $\breve{u}_{k,i}(\varepsilon,t)$ such that

$$\breve{u}_{k,i}(\varepsilon,t)\leqslant\hat{u}_{k,i}(\varepsilon,t)$$

and

$$\sum_{i=k+1}^{n}\breve{u}_{k,i}(\varepsilon,t)=\sum_{i=-n_0+1}^{k-1}\bar{u}_{i,k}(\varepsilon,t).$$

Let

$$\bar{u}_{k,i}(\varepsilon,t)=\begin{cases}\breve{u}_{k,i}(\varepsilon,t), & \text{if } t\in B_k^\varepsilon,\\ \hat{u}_{k,i}(\varepsilon,t), & \text{if } t\notin B_k^\varepsilon.\end{cases}$$

Then we get

$$\bar{u}_{i,j}(\varepsilon,t),\quad (i,j)\in A.$$

Step III. Set

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\boldsymbol{x}}(\varepsilon,t)=\mathcal{L}\big(\bar{\boldsymbol{u}}(\varepsilon,t),\boldsymbol{z}\big),\qquad \bar{\boldsymbol{x}}(\varepsilon,0)=\frac{\boldsymbol{x}}{a(H)}+(1,\dots,1)'\varepsilon^\delta.$$

For $j = m+1, \dots, n$, define

$$\widehat{B}_j^\varepsilon = \Big\{t\colon \big(H_j - \bar{x}_j(\varepsilon, t)\big) - \inf_{0 \leqslant s \leqslant t} \big\{H_j - \bar{x}_j(\varepsilon, s)\big\} = 0 \text{ and } \bar{x}_j(\varepsilon, t) > H_j\Big\}.$$

Note that for $t \in \widehat{B}_j^\varepsilon$,

$$\sum_{i=-n_0+1}^{j-1} \bar{u}_{i,j}(\varepsilon, t) > z_j.$$

For $j = m+1, \dots, n$ and $i = -n_0+1, \dots, j-1$, choose $\tilde{u}_{i,j}(\varepsilon, t)$ such that

$$\tilde{u}_{i,j}(\varepsilon, t) \leqslant \bar{u}_{i,j}(\varepsilon, t), \qquad \sum_{i=-n_0+1}^{j-1} \tilde{u}_{i,j}(\varepsilon, t) = z_j.$$

Define

$$u_{i,j}(\varepsilon, t) = \begin{cases} \bar{u}_{i,j}(\varepsilon, t), & t \notin \widehat{B}_j^\varepsilon, \\ \tilde{u}_{i,j}(\varepsilon, t), & t \in \widehat{B}_j^\varepsilon. \end{cases}$$

Sub-step k ($k = m, \dots, 1$): Set

$$\widehat{B}_k^\varepsilon = \Big\{t\colon \big(H_k - \tilde{x}_k^k(\varepsilon, t)\big) - \inf_{0 \leqslant s \leqslant t} \big\{H_k - \tilde{x}_k^k(\varepsilon, s) = 0\big\} \text{ and } \tilde{x}_k^k(\varepsilon, t) > H_k\Big\},$$

where

$$\tilde{x}_k^k(\varepsilon, t) = \frac{x_k}{a(H)} + \varepsilon^\delta + \int_0^t \left[\sum_{i=-n_0+1}^{k-1} \bar{u}_{i,k}(\varepsilon, s) - \sum_{i=k+1}^{n} u_{k,i}(\varepsilon, s)\right] \mathrm{d}s.$$

For $t \in \widehat{B}_k^\varepsilon$ and $i = -n_0+1, \dots, k-1$, choose $\tilde{u}_{i,k}(\varepsilon, t)$ such that

$$\tilde{u}_{i,k}(\varepsilon, t) \leqslant \bar{u}_{i,k}(\varepsilon, t), \qquad \sum_{i=-n_0+1}^{k-1} \tilde{u}_{i,k}(\varepsilon, t) = \sum_{i=k+1}^{n} u_{k,i}(\varepsilon, t).$$

For $i = -n_0+1, \dots, k-1$, define

$$u_{i,k}(\varepsilon, t) = \begin{cases} \tilde{u}_{i,k}(\varepsilon, t), & \text{if } t \in \widehat{B}_k^\varepsilon, \\ \bar{u}_{i,k}(\varepsilon, t), & \text{if } t \notin \widehat{B}_k^\varepsilon. \end{cases}$$

Then we get

$$u_{i,j}(\varepsilon, t), \quad (i, j) \in A.$$

6. Concluding remarks

In this paper, we have considered a hierarchical production control of a jobshop with a discounted cost criterion. We have constructed near optimal control policy for the jobshop based on the corresponding limiting problem that is simpler than the original problem. The main advantage of our approach is the reduction of the system complexity. It would be of interest to consider hierarchical production controls for long-run average cost objective. This is a topic of future research.

Acknowledgments

The authors thank Associate Editor, Professor Jerzy Filar, and the reviewer for valuable suggestions.

Appendix A

In this appendix we provide proofs of Lemmas 4.4 and 4.5.

Proof of Lemma 4.4. First we estimate the probability that $\hat{x}_k(\varepsilon,t) \notin (\varepsilon^\delta/2, H_k - \varepsilon^\delta/2)$ ($k = 1,\dots,m$). By (41) and (42),

$$\begin{aligned}
&\Pr\left(\hat{x}_k(\varepsilon,t) \leqslant \frac{\varepsilon^\delta}{2} \text{ or } \hat{x}_k(\varepsilon,t) \geqslant H_k - \frac{\varepsilon^\delta}{2}\right) \\
&\quad \leqslant \Pr\left(\left|\hat{x}_k(\varepsilon,t) - x_k(\delta,\varepsilon,t)\right| \geqslant \frac{\varepsilon^\delta}{2}\right) \\
&\quad \leqslant \Pr\left(\sum_{\ell=1}^{p}\left|\int_0^t (I_{\{\boldsymbol{k}(\varepsilon,s)=\boldsymbol{k}^\ell\}} - \gamma_\ell)\left(\sum_{i=-n_0+1}^{k-1} u_{i,k}^\ell(\delta,\varepsilon,s) - \sum_{i=k+1}^{n} u_{k,i}^\ell(\delta,\varepsilon,s)\right)\mathrm{d}s\right| \geqslant \frac{\varepsilon^\delta}{2}\right).
\end{aligned}$$

Lemma 4.2 implies that there exist positive constants ε_{11} (< 1), κ_1, and C_1, such that for $k = 1,\dots,m$ and $\varepsilon \in (0,\varepsilon_{11})$,

$$\begin{aligned}
&\Pr\left(\hat{x}_k(\varepsilon,t) \leqslant \frac{\varepsilon^\delta}{2} \text{ or } \hat{x}_k(\varepsilon,t) \geqslant H_k - \frac{\varepsilon^\delta}{2}\right) \\
&\quad \leqslant C_1 \exp\left\{-\kappa_1 \varepsilon^{-(1/2-\delta)}(1+t)^{-3}\right\}.
\end{aligned} \tag{A.1}$$

Note that for $t \in [0, \varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}]$ with $\varepsilon \in (0,\varepsilon_{11})$,

$$\exp\left\{-\kappa_1 \varepsilon^{-(1/2-\delta)}(1+t)^{-3}\right\} \leqslant \exp\left\{-\frac{\kappa_1}{2^3}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}.$$

Thus, for $k = 1,\dots,m$ and $\varepsilon \in (0,\varepsilon_{11})$,

$$\begin{aligned}
&\int_0^\infty \exp\left\{\frac{-\rho t}{2^{2m+1}}\right\} \cdot \Pr\left(\hat{x}_k(\varepsilon,t) \leqslant \frac{\varepsilon^\delta}{2} \text{ or } \hat{x}_k(\varepsilon,t) \geqslant H_k - \frac{\varepsilon^\delta}{2}\right)\mathrm{d}t \\
&\quad \leqslant C_1 \int_0^\infty \exp\left\{-\left[\frac{\rho t}{2^{2m+1}} + \kappa_1 \varepsilon^{-(1/2-\delta)}(1+t)^{-3}\right]\right\}\mathrm{d}t \\
&\quad \leqslant C_1 \int_{\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}}^\infty \exp\left\{-\left[\frac{\rho t}{2^{2m+1}} + \kappa_1 \varepsilon^{-(1/2-\delta)}(1+t)^{-3}\right]\right\}\mathrm{d}t \\
&\qquad + C_1 \int_0^{\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}} \exp\left\{-\left[\frac{\rho t}{2^{2m+1}} + \kappa_1 \varepsilon^{-(1/2-\delta)}(1+t)^{-3}\right]\right\}\mathrm{d}t \\
&\quad \leqslant C_2 \exp\left\{-\kappa_{11} \varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\},
\end{aligned} \tag{A.2}$$

for some $C_2 > 0$ and $\kappa_{11} > 0$. In the same way, for $k = m+1, \dots, n$ and $\varepsilon \in (0, \varepsilon_{11})$,

$$\int_0^\infty \exp\left\{\frac{-\rho t}{2^{2m+1}}\right\} \cdot \Pr\left(\hat{x}_k(\varepsilon, t) \geqslant H_k - \frac{\varepsilon^\delta}{2}\right) \mathrm{d}t \leqslant C_2 \exp\left\{-\kappa_{11}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}. \tag{A.3}$$

Let

$$B_1^\varepsilon = \left\{t\colon\ \hat{x}_1(\varepsilon, t) - \inf_{0 \leqslant s \leqslant t} \hat{x}_1(\varepsilon, s) = 0 \text{ and } \hat{x}_1(\varepsilon, t) < 0\right\}.$$

Then, for $t \in B_1^\varepsilon$,

$$\sum_{i=-n_0+1}^{0} \hat{u}_{i,1}(\varepsilon, t) \leqslant \sum_{i=2}^{n} \hat{u}_{1,i}(\varepsilon, t). \tag{A.4}$$

For $t \in B_1^\varepsilon$ and $i = 2, \dots, n$, choose $\check{u}_{1,i}(\varepsilon, t)$ such that

$$\check{u}_{1,i}(\varepsilon, t) \leqslant \hat{u}_{1,i}(\varepsilon, t)$$

and

$$\sum_{i=2}^{n} \check{u}_{1,i}(\varepsilon, t) = \sum_{i=-n_0+1}^{0} \hat{u}_{i,1}(\varepsilon, t).$$

Define for $j = -n_0 + 1, \dots, 0$,

$$\bar{u}_{j,1}(\varepsilon, t) = \hat{u}_{j,1}(\varepsilon, t)$$

and for $i = 2, \dots, n$,

$$\bar{u}_{1,i}(\varepsilon, t) = \begin{cases} \hat{u}_{1,i}(\varepsilon, t), & t \notin B_1^\varepsilon, \\ \check{u}_{1,i}(\varepsilon, t), & t \in B_1^\varepsilon. \end{cases}$$

Thus,

$$\bar{x}_1(\varepsilon, t) = x_1(\delta) + \int_0^t \left[\sum_{j=-n+0+1}^{0} \bar{u}_{j,1}(\varepsilon, s) - \sum_{j=2}^{n} \bar{u}_{1,j}(\varepsilon, s)\right] \mathrm{d}s \geqslant 0.$$

Furthermore, for $i = 2, \dots, n$,

$$\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m+1}}\right\} \cdot \left|\bar{u}_{1,i}(\varepsilon, t) - \hat{u}_{1,i}(\varepsilon, t)\right| \mathrm{d}t \\
&\quad = \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m+1}}\right\} \cdot \left|\bar{u}_{1,i}(\varepsilon, t) - \hat{u}_{1,i}(\varepsilon, t)\right| I_{\{t \in B_1^\varepsilon\}} \mathrm{d}t \\
&\quad \leqslant \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m+1}}\right\} \cdot \left|\bar{u}_{1,i}(\varepsilon, t) - \hat{u}_{1,i}(\varepsilon, t)\right| I_{\{\hat{x}_1(\varepsilon, t) < 0\}} \mathrm{d}t \\
&\quad \leqslant \max_{\substack{1 \leqslant \ell \leqslant p \\ 1 \leqslant r \leqslant m_c}} \{k_r^\ell\} \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m+1}}\right\} \cdot I_{\{\hat{x}_1(\varepsilon, t) < 0\}} \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \max_{\substack{1\leqslant \ell\leqslant p\\ 1\leqslant r\leqslant m_c}} \{k_r^\ell\} \int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m+1}}\right\} \cdot \Pr\big(\hat{x}_1(\varepsilon,t)\leqslant 0\big)\,\mathsf{d}t \\
&\leqslant \max_{\substack{1\leqslant \ell\leqslant p\\ 1\leqslant r\leqslant m_c}} \{k_r^\ell\} \int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m+1}}\right\} \cdot \Pr\left(\hat{x}_1^\varepsilon(t)\leqslant \frac{\varepsilon^\delta}{2}\right)\mathsf{d}t \\
&= C_3 \exp\big\{-\kappa_{11}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\big\} \quad \big(\text{by (A.2)}\big),
\end{aligned} \tag{A.5}
$$

for some $C_3 > 0$ and $\varepsilon \in (0, \varepsilon_{11})$. Note that

$$
\begin{aligned}
&\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\bar{x}_1(\varepsilon,t)\geqslant H_1 - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta\right)\mathsf{d}t \\
&\quad\leqslant \int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\hat{x}_1(\varepsilon,t)\geqslant H_1 - \frac{1}{2}\varepsilon^\delta\right)\mathsf{d}t \\
&\qquad + \int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\bar{x}_1(\varepsilon,t)\geqslant H_1 - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta \text{ and } \hat{x}_1(\varepsilon,t)\leqslant H_1 - \frac{1}{2}\varepsilon^\delta\right)\mathsf{d}t \\
&\quad\leqslant \int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\hat{x}_1(\varepsilon,t)\geqslant H_1 - \frac{1}{2}\varepsilon^\delta\right)\mathsf{d}t \\
&\qquad + \int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\big|\bar{x}_1(\varepsilon,t) - \hat{x}_1(\varepsilon,t)\big| \geqslant \frac{1}{2^2}\varepsilon^\delta\right)\mathsf{d}t \\
&\quad\leqslant \int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\hat{x}_1(\varepsilon,t)\geqslant H_1 - \frac{1}{2}\varepsilon^\delta\right)\mathsf{d}t \\
&\qquad + \frac{2^2}{\varepsilon^\delta}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \mathsf{E}\int_0^t \left|\sum_{j=2}^n \bar{u}_{1,j}(\varepsilon,s) - \sum_{j=2}^n \hat{u}_{1,j}(\varepsilon,s)\right| \mathsf{d}s\,\mathsf{d}t.
\end{aligned}
$$

Using the method given in (50), we can show, in view of (A.2) and (A.5), that there are positive constants C_4, ε_{21} $(< \varepsilon_{11})$, and κ_{21} such that for $\varepsilon \in (0, \varepsilon_{21})$,

$$
\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\bar{x}_1(\varepsilon,t)\geqslant H_1 - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta\right)\mathsf{d}t \leqslant C_4 \exp\big\{-\kappa_{21}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\big\}. \tag{A.6}
$$

Now we consider the system

$$\begin{cases} \bar{x}_j^1(\varepsilon,t) = x_j(\delta) + \displaystyle\int_0^t \Bigg(\bar{u}_{1,j}(\varepsilon,s) + \sum_{i\neq 1,\, i=-n_0+1}^{j-1} \hat{u}_{i,j}(\varepsilon,s) - \sum_{i=j+1}^{n} \hat{u}_{j,i}(\varepsilon,s) \Bigg)\, \mathrm{d}s, \\ \quad j = 2,\dots,m, \\ \bar{x}_j^1(\varepsilon,t) = x_j(\delta) + \displaystyle\int_0^t \Bigg(\bar{u}_{1,j}(\varepsilon,s) + \sum_{i\neq 1,\, i=-n_0+1}^{m} \hat{u}_{i,j}(\varepsilon,s) - z_j \Bigg)\, \mathrm{d}s, \\ \quad j = m+1,\dots,n. \end{cases} \tag{A.7}$$

Note that

$$\begin{aligned} &\Pr\left(\bar{x}_2^1(\varepsilon,t) \notin \left(\left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta,\, H_2 - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta\right)\right) \\ &\quad \leqslant \Pr\left(\hat{x}_2(\varepsilon,t) \leqslant \frac{\varepsilon^\delta}{2} \text{ or } \hat{x}_2(\varepsilon,t) \geqslant H_2 - \frac{\varepsilon^\delta}{2}\right) \\ &\qquad + \Pr\left(\left\{\bar{x}_2^1(\varepsilon,t) \leqslant \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta \text{ or } \bar{x}_2^1(\varepsilon,t) \geqslant H_2 - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta\right\}\right. \\ &\qquad \left. \cap \left\{\hat{x}_2(\varepsilon,t) \in \left(\frac{\varepsilon^\delta}{2},\, H_2 - \frac{\varepsilon^\delta}{2}\right)\right\}\right) \\ &\quad \leqslant \Pr\left(\hat{x}_2(\varepsilon,t) \leqslant \frac{\varepsilon^\delta}{2} \text{ or } \hat{x}_2(\varepsilon,t) \geqslant H_2 - \frac{\varepsilon^\delta}{2}\right) \\ &\qquad + \Pr\left(\left|\bar{x}_2^1(\varepsilon,t) - \hat{x}_2(\varepsilon,t)\right| \geqslant \frac{\varepsilon^\delta}{2^2}\right). \end{aligned}$$

Then, by (A.2), similar to (A.6), there are $C_5 > 0$, $\varepsilon_{31} > 0$, and $\kappa_{31} > 0$, such that for $\varepsilon \in (0, \varepsilon_{31})$,

$$\begin{aligned} &\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\bar{x}_2^1(\varepsilon,t) \notin \left(\left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta,\, H_2 - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta\right)\right) \mathrm{d}t \\ &\quad \leqslant C_5 \exp\left\{-\kappa_{31}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}. \end{aligned} \tag{A.8}$$

In the same way, for $j = 3,\dots,m$ and $\varepsilon \in (0, \varepsilon_{31})$,

$$\begin{aligned} &\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\bar{x}_j^1(\varepsilon,t) \notin \left(\left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta,\, H_j - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta\right)\right) \mathrm{d}t \\ &\quad \leqslant C_5 \exp\left\{-\kappa_{31}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}, \end{aligned} \tag{A.9}$$

and for $j = m+1,\dots,n$ and $\varepsilon \in (0, \varepsilon_{31})$,

$$\begin{aligned} &\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m}}\right\} \cdot \Pr\left(\bar{x}_j^1(\varepsilon,t) \notin \left(-\infty,\, H_j - \left(\frac{1}{2}-\frac{1}{2^2}\right)\varepsilon^\delta\right)\right) \mathrm{d}t \\ &\quad \leqslant C_5 \exp\left\{-\kappa_{31}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}. \end{aligned} \tag{A.10}$$

Repeating the procedure of modifying $\{\hat{u}_{2,j}(\varepsilon,t), j=3,\dots,n\}$ and $\{\hat{u}_{j,2}(\varepsilon,t), j=-n_0+1,\dots,2,0\}$ on the system (A.7), suppose that the following modifications

$$\begin{aligned}&\hat{u}_{i,j}(\varepsilon,t),\quad -n_0+1\leqslant i\leqslant j-1,\\&\hat{u}_{j,i}(\varepsilon,t),\quad i=j+1,\dots,n,\end{aligned}$$

for $j=1,\dots,r$ $(1\leqslant r<m)$ have been done. That is, we get that for $j=1,\dots,r$,

$$\begin{aligned}&\bar{u}_{i,j}(\varepsilon,t),\quad -n_0+1\leqslant i\leqslant j-1,\\&\bar{u}_{j,\ell}(\varepsilon,t),\quad \ell=j+1,\dots,n,\end{aligned}$$

such that for $j=1,\dots,r$,

$$\begin{aligned}&\bar{u}_{i,j}(\varepsilon,t)\leqslant\hat{u}_{i,j}(\varepsilon,t),\quad -n_0+1\leqslant i\leqslant j-1,\\&\bar{u}_{j,i}(\varepsilon,t)\leqslant\hat{u}_{j,i}(\varepsilon,t),\quad i=j+1,\dots,n,\end{aligned}$$

and

$$\begin{cases}\bar{x}_j(\varepsilon,t)=x_j(\delta)+\displaystyle\int_0^t\Bigg[\sum_{\ell=-n_0+1}^{j-1}\bar{u}_{\ell,j}(\varepsilon,s)-\sum_{\ell=j+1}^{n}\bar{u}_{j,\ell}(\varepsilon,s)\Bigg]\mathrm{d}s,\quad j=1,\dots,r,\\[2ex] \bar{x}^r_j(\varepsilon,t)=x_j(\delta)+\displaystyle\int_0^t\Bigg[\sum_{\ell=-n_0+1}^{0}\hat{u}_{\ell,j}(\varepsilon,s)+\sum_{\ell=1}^{r}\bar{u}_{\ell,j}(\varepsilon,s)+\sum_{\ell=r+1}^{j-1}\hat{u}_{\ell,j}(\varepsilon,s)\\ \qquad\qquad\displaystyle-\sum_{\ell=j+1}^{n}\hat{u}_{j,\ell}(\varepsilon,s)\Bigg]\mathrm{d}s,\quad j=r+1,\dots,m,\\[2ex] \bar{x}^r_j(\varepsilon,t)=x_j(\delta)+\displaystyle\int_0^t\Bigg[\sum_{\ell=-n_0+1}^{0}\hat{u}_{\ell,j}(\varepsilon,s)+\sum_{\ell=1}^{r}\bar{u}_{\ell,j}(\varepsilon,s)+\sum_{\ell=r+1}^{m}\hat{u}_{\ell,j}(\varepsilon,s)-z_j\Bigg]\mathrm{d}s,\\ \quad j=m+1,\dots,n.\end{cases}\tag{A.11}$$

Furthermore, there are positive constants C_6, ε_{41} $(<\varepsilon_{31})$, and κ_{41}, such that for $j=1,\dots,r$ and $\varepsilon\in(0,\varepsilon_{41})$,

$$\mathsf{E}\int_0^\infty\exp\left\{-\frac{\rho t}{2^{2m-r+2}}\right\}\cdot\left|\bar{u}_{\ell,j}(\varepsilon,t)-\hat{u}_{\ell,j}(\varepsilon,t)\right|\mathrm{d}t\leqslant C_6\exp\left\{-\kappa_{41}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\},$$
$$\ell=-n_0+1,\dots,j-1,\tag{A.12}$$

$$\mathsf{E}\int_0^\infty\exp\left\{-\frac{\rho t}{2^{2m-r+2}}\right\}\cdot\left|\bar{u}_{j,\ell}(\varepsilon,t)-\hat{u}_{j,\ell}(\varepsilon,t)\right|\mathrm{d}t\leqslant C_6\exp\left\{-\kappa_{41}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\},$$
$$\ell=j+1,\dots,n,\tag{A.13}$$

$$\bar{x}_j(\varepsilon,t)\geqslant 0,\tag{A.14}$$

$$\mathsf{E}\int_0^\infty\exp\left\{-\frac{\rho t}{2^{2m-r+1}}\right\}\cdot\Pr\left(\bar{x}_j(\varepsilon,t)\geqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^{r}\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right)\mathrm{d}t$$
$$\leqslant C_6\exp\left\{-\kappa_{41}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\},\tag{A.15}$$

for $j = r+1, \ldots, m$,

$$\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r+1}}\right\} \\
&\quad \cdot \Pr\left\{\bar{x}_j^r(\varepsilon,t) \notin \left(\left[\frac{1}{2}-\sum_{i=1}^r \frac{1}{2^{i+1}}\right]\varepsilon^\delta, H_j - \left[\frac{1}{2}-\sum_{i=1}^r \frac{1}{2^{i+1}}\right]\varepsilon^\delta\right)\right\} \mathrm{d}t \\
&\leqslant C_6 \exp\left\{-\kappa_{41}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}, \qquad\qquad (A.16)
\end{aligned}$$

and for $j = m+1, \ldots, n$,

$$\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r+1}}\right\} \cdot \Pr\left\{\bar{x}_j^r(\varepsilon,t) \notin \left(-\infty, H_j - \left[\frac{1}{2}-\sum_{i=1}^r \frac{1}{2^{i+1}}\right]\varepsilon^\delta\right)\right\} \mathrm{d}t \\
&\leqslant C_6 \exp\left\{-\kappa_{41}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}. \qquad\qquad (A.17)
\end{aligned}$$

Now we modify

$$\begin{aligned}
&\hat{u}_{\ell,r+1}(\varepsilon,t), \quad \ell = -n_0+1, \ldots, 0, \\
&\hat{u}_{r+1,\ell}(\varepsilon,t), \quad \ell = r+2, \ldots, n,
\end{aligned}$$

for the system (A.11). To do this, we define

$$B_{r+1}^\varepsilon = \left\{t\colon \bar{x}_{r+1}^r(\varepsilon,t) - \inf_{0\leqslant s\leqslant t} \bar{x}_{r+1}^r(\varepsilon,s) = 0 \text{ and } \bar{x}_{r+1}^r(\varepsilon,t) < 0\right\}.$$

Then, we have that for $t \in B_{r+1}^\varepsilon$,

$$\sum_{\ell=r+2}^n \hat{u}_{r+1,\ell}(\varepsilon,t) \geqslant \sum_{\ell=-n_0+1}^0 \hat{u}_{\ell,r+1}(\varepsilon,t) + \sum_{\ell=1}^r \bar{u}_{\ell,r+1}(\varepsilon,t). \qquad (A.18)$$

For $\ell = r+2, \ldots, n$ and $t \in B_{r+1}^\varepsilon$, we choose $\check{u}_{r+1,\ell}(\varepsilon,t)$ such that

$$\check{u}_{r+1,\ell}(\varepsilon,t) \leqslant \hat{u}_{r+1,\ell}(\varepsilon,t)$$

and

$$\sum_{\ell=r+2}^n \check{u}_{r+1,\ell}(\varepsilon,t) = \sum_{\ell=-n_0+1}^0 \hat{u}_{\ell,r+1}(\varepsilon,t) + \sum_{\ell=1}^r \bar{u}_{\ell,r+1}(\varepsilon,t).$$

Define

$$\begin{aligned}
&\bar{u}_{i,r+1}(\varepsilon,t) = \hat{u}_{i,r+1}(\varepsilon,t), \quad i = -n_0+1, \ldots, 0, \\
&\bar{u}_{r+1,\ell}(\varepsilon,t) = \begin{cases} \hat{u}_{r+1,\ell}(\varepsilon,t), & \text{if } t \notin B_{r+1}^\varepsilon, \\ \check{u}_{r+1,\ell}(\varepsilon,t), & \text{if } t \in B_{r+1}^\varepsilon. \end{cases}
\end{aligned}$$

By the definition of B_{r+1}^ε, we know that

$$\bar{x}_{r+1}(\varepsilon,t) = x_{r+1}(\delta) + \int_0^t \left[\sum_{\ell=-n_0+1}^r \bar{u}_{\ell,r+1}(\varepsilon,s) - \sum_{\ell=r+2}^n \bar{u}_{r+1,\ell}(\varepsilon,s)\right] \mathrm{d}s \geqslant 0.$$

Furthermore, for $\ell = r+2, \ldots, n$,

$$
\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r+1}}\right\} \cdot \left|\bar{u}_{r+1,\ell}(\varepsilon,t) - \hat{u}_{r+1,\ell}(\varepsilon,t)\right| \mathrm{d}t \\
&\quad \leqslant \max_{\substack{1\leqslant \ell \leqslant p\\ 1\leqslant r\leqslant m_c}} \left\{k_r^\ell\right\} \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r+1}}\right\} \cdot \Pr\left(\bar{x}_{r+1}^r(\varepsilon,t) \leqslant 0\right) \mathrm{d}t \\
&\quad \leqslant \max_{\substack{1\leqslant \ell \leqslant p\\ 1\leqslant r\leqslant m_c}} \left\{k_r^\ell\right\} \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r+1}}\right\} \cdot \Pr\left(\bar{x}_{r+1}^r(\varepsilon,t) \leqslant \left(\frac{1}{2} - \sum_{j=1}^{r} \frac{1}{2^{j+1}}\right)\varepsilon^\delta\right) \mathrm{d}t.
\end{aligned}
$$

Consequently, by (A.16) with $j = r+1$, for $\varepsilon \in (0, \varepsilon_{41})$,

$$
\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r+1}}\right\} \cdot \left|\bar{u}_{r+1,\ell}(\varepsilon,t) - \hat{u}_{r+1,\ell}(\varepsilon,t)\right| \mathrm{d}t \\
&\quad \leqslant C_7 \exp\left\{-\kappa_{51}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\},
\end{aligned}
\tag{A.19}
$$

for some $C_7 > 0$ and $\kappa_{51} > 0$. Similar to (A.6), we use (A.12) and (A.15) to obtain that there are $\varepsilon_{51} > 0$ $(< \varepsilon_{41})$ and $C_8 > 0$, such that for $\varepsilon \in (0, \varepsilon_{51})$,

$$
\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r}}\right\} \cdot \Pr\left(\bar{x}_{r+1}(\varepsilon,t) \geqslant H_{r+1} - \left(\frac{1}{2} - \sum_{j=1}^{r+1} \frac{1}{2^{j+1}}\right)\varepsilon^\delta\right) \mathrm{d}t \\
&\quad \leqslant C_8 \exp\left\{-\kappa_{51}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}.
\end{aligned}
\tag{A.20}
$$

For $j = r+2, \ldots, m$, let

$$
\begin{aligned}
\bar{x}_j^{r+1}(\varepsilon,t) = x_j(\delta) + \int_0^t \Bigg[&\sum_{\ell=-n_0+1}^{0} \hat{u}_{\ell,j}(\varepsilon,s) + \sum_{\ell=1}^{r+1} \bar{u}_{\ell,j}(\varepsilon,s) + \sum_{\ell=r+2}^{j-1} \hat{u}_{\ell,j}(\varepsilon,s). \\
&- \sum_{\ell=j+1}^{n} \hat{u}_{j,\ell}(\varepsilon,s)\Bigg] \mathrm{d}s,
\end{aligned}
$$

and for $j = m+1, \ldots, n$, let

$$
\bar{x}_j^{r+1}(\varepsilon,t) = x_j(\delta) + \int_0^t \left[\sum_{\ell=-n_0+1}^{0} \hat{u}_{\ell,j}(\varepsilon,s) + \sum_{\ell=1}^{r+1} \bar{u}_{\ell,j}(\varepsilon,s) + \sum_{\ell=r+2}^{m} \hat{u}_{\ell,j}(\varepsilon,s) - z_j \right] \mathrm{d}s.
$$

Similar to (A.8), we can see from (A.12)–(A.15) that there are positive constants C_9, ε_{61} $(< \varepsilon_{51})$, and κ_{61}, such that for $j = r+2, \ldots, m$ and $\varepsilon \in (0, \varepsilon_{61})$,

$$
\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{2m-r}}\right\}
$$

$$
\begin{aligned}
&\cdot \Pr\left\{\bar{x}_j^{r+1}(\varepsilon,t) \notin \left(\left[\frac{1}{2}-\sum_{i=1}^{r+1}\frac{1}{2^{i+1}}\right]\varepsilon^{\delta}, H_j-\left[\frac{1}{2}-\sum_{i=1}^{r+1}\frac{1}{2^{i+1}}\right]\varepsilon^{\delta}\right)\right\} \mathrm{d}t \\
&\leqslant C_9 \exp\left\{-\kappa_{61}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}, \qquad \text{(A.21)}
\end{aligned}
$$

and for $j = m+1, \ldots, n$,

$$
\begin{aligned}
&\mathsf{E}\int_0^{\infty} \exp\left\{-\frac{\rho t}{2^{2m-r}}\right\} \cdot \Pr\left\{\bar{x}_j^{r+1}(\varepsilon,t) \notin \left(-\infty, H_j-\left[\frac{1}{2}-\sum_{i=1}^{r+1}\frac{1}{2^{i+1}}\right]\varepsilon^{\delta}\right)\right\} \mathrm{d}t \\
&\leqslant C_9 \exp\left\{-\kappa_{61}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}. \qquad \text{(A.22)}
\end{aligned}
$$

When $r = m-1$ we get Lemma 4.4. $\square$

Proof of Lemma 4.5. For $j = m+1, \ldots, n$, let

$$
\widehat{B}_j^{\varepsilon} = \left\{t\colon \left(H_j - \bar{x}_j(\varepsilon,t)\right) - \inf_{0\leqslant s\leqslant t}\left\{H_j - \bar{x}_j(\varepsilon,s)\right\} = 0 \text{ and } \bar{x}_j(\varepsilon,t) > H_j\right\}.
$$

Then for $t \in \widehat{B}_j^{\varepsilon}$,

$$
\sum_{\ell=-n_0+1}^{j-1} \bar{u}_{\ell,j}(\varepsilon,t) \geqslant z_j. \qquad \text{(A.23)}
$$

For $\ell = -n_0+1, \ldots, m$, choose

$$
\breve{u}_{\ell,j}(\varepsilon,t) \leqslant \bar{u}_{\ell,j}(\varepsilon,t)
$$

such that

$$
\sum_{\ell=-n_0+1}^{m} \breve{u}_{\ell,j}(\varepsilon,t) = z_j.
$$

Define

$$
u_{\ell,j}(\varepsilon,t) = \begin{cases} \bar{u}_{\ell,j}(\varepsilon,t), & t \notin \widehat{B}_j^{\varepsilon}, \\ \breve{u}_{\ell,j}(\varepsilon,t), & t \in \widehat{B}_j^{\varepsilon}. \end{cases}
$$

Then

$$
x_j(\varepsilon,t) = x_j(\delta) + \int_0^t \left[\sum_{\ell=-n_0+1}^{m} u_{\ell,j}(\varepsilon,s) - z_j\right] \mathrm{d}s \leqslant H_j.
$$

Furthermore, for $j = m+1, \ldots, n$ and $\ell = -n_0+1, \ldots, m$,

$$
\begin{aligned}
&\mathsf{E}\int_0^{\infty} \exp\left\{-\frac{\rho t}{2^{m+1}}\right\} \cdot \left|u_{\ell,j}(\varepsilon,t) - \bar{u}_{\ell,j}(\varepsilon,t)\right| \mathrm{d}t \\
&\leqslant \max_{\substack{1\leqslant \ell\leqslant p \\ 1\leqslant r\leqslant m_c}} \left\{k_r^{\ell}\right\} \mathsf{E}\int_0^{\infty} \exp\left\{-\frac{\rho t}{2^{m+1}}\right\} \cdot \Pr\left(\bar{x}_j(\varepsilon,t) > H_j\right) \mathrm{d}t
\end{aligned}
$$

$$\leqslant \max_{\substack{1\leqslant \ell\leqslant p\\ 1\leqslant r\leqslant m_c}} \{k_r^\ell\} \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{m+1}}\right\}\cdot \Pr\left(\bar{x}_j(\varepsilon,t)\geqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^m\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right)\mathrm{d}t.$$

Using (46), we get that there exist positive constants C_1, κ_{12}, and ε_{12}, such that for $j = m+1,\dots,n$ and $\varepsilon\in(0,\varepsilon_{12})$,

$$\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{m+1}}\right\}\cdot \left|u_{\ell,j}(\varepsilon,t)-\bar{u}_{\ell,j}(\varepsilon,t)\right|\mathrm{d}t\leqslant C_1\exp\left\{-\kappa_{12}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}. \tag{A.24}$$

For $j=1,\dots,m$,

$$\hat{x}_j^{m-1}(\varepsilon,t)=x_j(\delta)+\int_0^t\left(\sum_{\ell=-n_0+1}^{j-1}\bar{u}_{\ell,j}(\varepsilon,s)-\sum_{\ell=j+1}^m\bar{u}_{j,\ell}(\varepsilon,s)-\sum_{\ell=m+1}^n u_{j,\ell}(\varepsilon,s)\right)\mathrm{d}s.$$

We know that $\hat{x}_j^{m-1}(t)\geqslant 0$. Furthermore, for $j=1,\dots,m$,

$$\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^m}\right\}\cdot \Pr\left(\hat{x}_j^{m-1}(\varepsilon,t)\geqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^{m+1}\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right)\mathrm{d}t\\
&\quad\leqslant \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^m}\right\}\cdot \Pr\left(\bar{x}_j(\varepsilon,t)\geqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^{m}\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right)\mathrm{d}t\\
&\qquad+\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^m}\right\}\cdot \Pr\left(\hat{x}_j^{m-1}(\varepsilon,t)\geqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^{m+1}\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right.\\
&\qquad\qquad\left.\text{and } \bar{x}_j(\varepsilon,t)\leqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^{m}\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right)\mathrm{d}t\\
&\quad\leqslant \mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^m}\right\}\cdot \Pr\left(\bar{x}_j(\varepsilon,t)\geqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^{m}\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right)\mathrm{d}t\\
&\qquad+\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^m}\right\}\cdot \Pr\left(\left|\hat{x}_j^{m-1}(\varepsilon,t)-\bar{x}_j(\varepsilon,t)\right|\geqslant\frac{\varepsilon^\delta}{2^{m+1}}\right)\mathrm{d}t.
\end{aligned}$$

Similar to (A.8), we get that there exist positive constants C_2, κ_{22}, and ε_{22}, such that for $j = 1,\dots,m$ and $\varepsilon\in(0,\varepsilon_{22})$,

$$\begin{aligned}
&\mathsf{E}\int_0^\infty \exp\left\{-\frac{\rho t}{2^{m-1}}\right\}\cdot \Pr\left(\hat{x}_j^{m-1}(\varepsilon,t)\geqslant H_j-\left(\frac{1}{2}-\sum_{i=1}^{m+1}\frac{1}{2^{i+1}}\right)\varepsilon^\delta\right)\mathrm{d}t\\
&\quad\leqslant C_2\exp\left\{-\kappa_{22}\varepsilon^{-\frac{1}{4}(\frac{1}{2}-\delta)}\right\}.
\end{aligned} \tag{A.25}$$

Now we consider the system

$$\begin{cases} \hat{x}_j^{m-1}(\varepsilon,t) = x_j(\delta) + \displaystyle\int_0^t \left[\sum_{\ell=-n_0+1}^{j-1} \bar{u}_{\ell,j}(\varepsilon,s) - \sum_{\ell=j+1}^{m} \bar{u}_{j,\ell}(\varepsilon,s) - \sum_{\ell=m+1}^{n} u_{j,\ell}(\varepsilon,s) \right] \mathrm{d}s, \\ \quad j = 1,\ldots,m-1, \\ \hat{x}_m^{m-1}(\varepsilon,t) = x_m(\delta) + \displaystyle\int_0^t \left[\sum_{\ell=-n_0+1}^{m-1} \bar{u}_{\ell,m}(\varepsilon,s) - \sum_{\ell=m+1}^{n} u_{m,\ell}(\varepsilon,s) \right] \mathrm{d}s. \end{cases}$$

By repeating the above procedure, we get Lemma 4.5. □

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